# Supersymmetry and Extra Dimensions: Example Sheet 4

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**Exercise 4.1:** Consider the Schrödinger equation for a particle moving in two dimensions $x$ and $y$. The second dimension is a circle or radius $r$. The potential corresponds to a square well ($V(x) = 0$ for $x \in (0, a)$ and $V = \infty$ otherwise). Derive the energy levels for the two-dimensional Schrödinger equation and compare the result with the standard one-dimensional situation in the limit $r \ll a$.

**Exercise 4.2:** Consider the following Lagrangian

$$\mathcal{S} \;=\; \int d^4x \, \left( \frac{1}{g^2} \, H_{\mu\nu\rho} \, H^{\mu\nu\rho} \;+\; a \, \epsilon^{\mu\nu\rho\sigma} \, \partial_\mu H_{\nu\rho\sigma} \right) \, .$$

Solve the equation of motion for the Lagrange multiplier $a$ to obtain an action for a propagating massless Kalb-Ramond field $B_{\mu\nu}$. Alternatively, solve the equation of motion for the field $H_{\nu\rho\sigma}$, to obtain an action for the propagating axion field $a$. What happens to the coupling $g$ under this transformation? Generalise your result for arbitrary dimensions and ranks of the tensors.

**Exercise 4.3:** Consider a massive antisymmetric tensor of rank $q$ in $D$ dimensions. Write up its Lagrangian up-to second derivatives. Describe a general Lagrangian that can reproduce the original Lagrangian and its dual. Determine the degrees of freedom of the original and dual tensors. Interpret this dualisation in terms of a functional Fourier transform. Can this also be used in the massless case?

**Exercise 4.4** On spacetimes with Lorentzian signature show that only in dimensions $D = 4k + 2$ there can be self-dual antisymmetric tensors. How many degrees of freedom do they have? What kind of $p$-branes they couple to? Explain the difference, if any, with Euclidean spaces.

**Exercise 4.5:** Show that the Kaluza-Klein dimensional reduction from $D = 5$ to $D = 4$ follows from a pure gravitational theory in five-dimensions, using ${}^{(5)}R = {}^{(4)}R - 2e^{-\sigma}\nabla^2 e^{\sigma} - \frac{1}{4}e^{2\sigma}F_{\mu\nu}F^{\mu\nu}$ where $G_{55} = e^{2\sigma}$. Relate the gauge transformation to the $U(1)$ isometry of the compact space.

**Exercise 4.6** Demonstrate that the volume of a $N - 1$ sphere of radius $r$ is

$$V_{N-1} = \frac{2\pi^{N/2}}{\Gamma(N/2)} \, r^{N-1} \tag{1}$$

*Hint: It may help to consider the integral* $I_N = \int d^N x e^{-\rho^2}$ *with* $\rho^2 = \sum_{i=1}^N x_i^2$. Use this result to derive an expression for the electric (and gravitational) potential in $D$ dimensions. Show that the potential due to a point particle in five dimensions reduces to the 4-dimensional potential at distances much larger than the size of the fifth dimension.

**Exercise 4.7:** Consider a five dimensional gravity theory with a negative cosmological constant $\Lambda < 0$, compactified on an interval $(0, \pi)$. Each end of the interval corresponds to a '3-brane' which we choose to have tension $\pm\Lambda/k$ respectively. Here $k$ is a common scale to be determined later in terms of the fundamental scale in 5D $M_*$ and $\Lambda$. Verify that the warped metric

$$\mathrm{d}s^2 \;=\; e^{-2A(\theta)}\,\eta_{\mu\nu}\,\mathrm{d}x^\mu\,\mathrm{d}x^\nu \;-\; r^2\,\mathrm{d}\theta^2$$

satisfies Einstein's equations. Here $W = e^{-2A(\theta)}$ is the warp factor and $r$ is a constant measuring the size of the interval. You can use that Einstein's equations reduce to

$$\frac{6\,A'^2}{r^2} \;=\; -\,\frac{\Lambda}{2M_*^3}\,, \qquad \frac{3\,A''}{r^2} \;=\; \frac{\Lambda}{2\,M_*^3\,kr}\,\left[\delta(\theta-\pi) \;-\; \delta(\theta)\right].$$

Solve for $A(\theta)$ and use the warp factor to show that the effective $4D$ Planck scale is now

$$M_{\rm pl}^2 \;=\; M_*^3\,r\int_{-\pi}^{\pi}\mathrm{d}\theta\; e^{-2A} \;=\; \frac{M_*^3}{k}\,\left(1 \;-\; e^{-2kr}\right).$$

Find the value of the constant $k$. Consider the Higgs Lagrangian on the brane at $\theta = \pi$, bring it into canonical form and show that the mass is proportional to the factor $e^{-k\pi r}$. How large can $r$ be in order to reproduce the electroweak scale from the Planck scale? Does this solve the hierarchy problem? How does the Planck scale differ from the 5D scale $M_*$?

**Exercise 4.8:** Imagine that it were possible to have particles with all possible spins up to $j = 3$. What would the maximum dimensionality of spacetime be consistent with supersymmetry?

**Exercise 4.9:** Starting with the field contents of IIA and IIB supergravities in $D = 10$ perform the dimensional reduction to $D = 9$ and count the number of degrees of freedom for each multiplet. Is the spectrum chiral? Perform directly the reduction from $D = 11$ to $D = 9$ and compare. Perform dimensional reduction of $IIB$ supergravity in $D = 10$ all the way to $D = 4$ and compare the number of degrees of freedom.

**Exercise 4.10:** Consider $\mathcal{N} = 1$ supergravity with three chiral superfields $S$, $T$, and $C$. In Planck units, the Kähler potential and superpotential are given by

$$\begin{aligned} K &= -\log\left(S + S^*\right) \;-\; 3\,\log\left(T + T^* - CC^*\right) \\ W &= C^3 \;+\; a\,e^{-\alpha S} \;+\; b\,, \end{aligned}$$

where $a$, $b$ are arbitrary complex numbers and $\alpha > 0$. Compute the scalar potential. Find the auxiliary field for $S, T, C$ and verify that supersymmetry is broken. Assuming that $C$ denotes a matter field with vanishing vev, find a minimum of the potential. Are there flat directions? A typical Kähler potential derived from string compactifications takes the form

$$K = -3\log\Gamma(\tau_i) \tag{2}$$

where $\Gamma$ is a homogeneous function of degree one of moduli fields $\tau_i$. Using the homogeneity equations $\tau_i\Gamma_i = \Gamma$ and $\tau_i\Gamma_{ij} = 0$ (where $\Gamma_i = \partial\Gamma/\partial\tau_i$, etc. ) show that

$$\tau_i K_{ij} = 3\Gamma_j/\Gamma\,, \qquad \Gamma_i K_{ij}^{-1}\Gamma_j/\Gamma^2 = 1/3$$

and deduce from this that if the superpotential does not depend on the $\tau_i$ fields then the corresponding contribution to the $N = 1$ supergravity scalar potential $V$ vanishes.