Mathematical Tripos Part III Dr Maciej Dunajski

GENERAL RELATIVITY: Examples 2 Michaelmas 2017

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1. Consider a change of basis $e'_\mu = (A^{-1})^\nu{}_\mu e_\nu$. Show that the components of a connection in the new basis are related to its components in the old basis by

$$\Gamma'^\mu_{\nu\rho} = A^\mu{}_\tau (A^{-1})^\lambda{}_\nu (A^{-1})^\sigma{}_\rho \Gamma^\tau_{\lambda\sigma} + A^\mu{}_\tau (A^{-1})^\sigma{}_\rho e_\sigma \left((A^{-1})^\tau{}_\nu\right) .$$

2. Let ∇ be the covariant derivative associated with a connection that is not torsion free. Let $T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$ where X and Y are vector fields. Show that this defines a $\binom{1}{2}$ tensor field T. This is called the *Torsion tensor*. Show that $2\nabla_{[a}\nabla_{b]} f = -T^c{}_{ab}\nabla_c f$, where f is any function.

3. Show that the equation of motion for a charged particle in a curved spacetime with electromagntic field tensor F_{ab} implies that $g_{ab}u^a u^b$ is constant along a charged-particle path with 4-velocity u^a. Hence this equation is consistent with the condition $g_{ab}u^a u^b = -1$ arising in the definition of 4-velocity.

4. Physically reasonable matter with energy-momentum tensor T^{ab} is expected to satisfy the *weak energy condition*, i.e.

$$T_{ab}u^a u^b \geq 0 \,,$$

for all timelike u^a. Give a physical interpretation for this condition. You measure the components of $T^a{}_b$ in some basis and determine its eigenvalues λ and eigenvectors v^a satisfying

$$T^a{}_b v^b = \lambda v^a \,.$$

You find that it has precisely one timelike eigenvector with eigenvalue $-\rho$ and three linear independent spacelike eigenvectors with eigenvalues $p_{(\alpha)}$. Under which necessary and sufficient condition on these eigenvalues is the weak energy condition satisfied?

5. (a) A scalar field obeying the Klein-Gordon equation $\nabla^a\nabla_a\Phi - m^2\Phi = 0$ has energy momentum tensor $T_{ab} = (\nabla_a\Phi)(\nabla_b\Phi) - (1/2)g_{ab}[(\nabla^c\Phi)(\nabla_c\Phi) + m^2\Phi^2]$. Show that T_{ab} is conserved.

(b) Show that the Maxwell equations imply that the energy momentum tensor of the electromagnetic field is conserved.

(c) Show that conservation of the energy momentum tensor of a perfect fluid imples that

$$u^a\nabla_a\rho + (\rho + P)\nabla_a u^a = 0, \qquad (\rho + P)u^b\nabla_b u_a = -(g_{ab} + u_a u_b)\nabla^b P \,.$$

6. A vector field Y is parallely propagated (with respect to the Levi-Civita connection) along an affinely parametrized geodesic with tangent vector X in a Riemannian space. Show that the magnitudes of the vectors and the angle between them are constant along the geodesic.
Let Y be a unit vector on the unit sphere that is initially tangent to the line $\phi = 0$ at a point on the equator. It is then moved by parallel propagation first along the equator to the location $\phi = \phi_0$, from there along the line $\phi = \phi_0$ to the North pole, and then back along the line $\phi = 0$ to its original position. By how much has it changed, and why?

7. Let ∇ be the covariant derivative associated with a torsion free connection. Derive the analogue of the Ricci identity for a covector field η:

$$\nabla_c\nabla_d\eta_a - \nabla_d\nabla_c\eta_a = -R^b{}_{acd}\eta_b \,.$$

8. Starting from the Bianchi identity $R^a{}_{b[cd;e]} = 0$, deduce the contracted Bianchi identity $G^a{}_{b;a} = 0$, where

$$G_{ab} = R_{ab} - \frac{1}{2} R g_{ab} \, ,$$

is the Einstein tensor.

9. Consider the Schwarzschild metric defined on examples sheet 1 (exercise 9). You have calculated the Christoffel symbols in the coordinate basis defined by Schwarzschild coordinates. Now calculate the Riemann tensor in the same basis. Show that the Ricci tensor vanishes. Calculate the scalar $R_{abcd} R^{abcd}$.

10. How many independent components does the Riemann tensor associated with the Levi-Civita connection have in two, three and four dimensions? Show that in two dimensions

$$R_{abcd} = \frac{1}{2} R (g_{ac} g_{bd} - g_{ad} g_{bc}) \, .$$

Discuss Einstein's theory in two dimensions.

11. In a spacetime of n dimensions define a tensor

$$C_{abcd} = R_{abcd} + \alpha (R_{ac} g_{bd} + R_{bd} g_{ac} - R_{ad} g_{bc} - R_{bc} g_{ad}) + \beta R (g_{ac} g_{bd} - g_{ad} g_{bc}) \, ,$$

where α and β are constants. Show that C_{abcd} has the same symmetries as R_{abcd}.

How do the coefficients α and β have to be chosen to set $C^a{}_{bad} = 0$? With this extra condition, C_{abcd} is called the *Weyl tensor*. Show that it vanishes if $n = 2$, 3.

Setting $n = 4$, how many independent components do R_{ab} and C_{abcd} have? What does the Weyl curvature represent physically? Show that in vacuum

$$\nabla^a C_{abcd} = 0 \, .$$

12. Use the Bianchi identity to derive the *Penrose equation* for a vacuum spacetime

$$\nabla^e \nabla_e R_{abcd} = 2 R_{aedf} R_b{}^e{}_c{}^f - 2 R_{aecf} R_b{}^e{}_d{}^f - R_{abef} R_{cd}{}^{ef} \, .$$