Mathematical Tripos Part III H.S. Reall

GENERAL RELATIVITY: Examples 4 Michaelmas 2016

1. Two bodies, with masses m_1, m_2, move in an elliptical Newtonian orbit with semi-major axis a and eccentricity e. Choosing coordinates so that the orbit lies in the (x, y) plane with the centre of mass at the origin, the bodies have positions $\mathbf{x_1} = (r_1 \cos\psi, r_1 \sin\psi, 0)$ and $\mathbf{x_2} = (-r_2 \cos\psi, -r_2 \sin\psi, 0)$ where

$$r_1 = \left(\frac{m_2}{m_1 + m_2}\right) r \qquad r_2 = \left(\frac{m_1}{m_1 + m_2}\right) r$$

The equation of the ellipse is

$$r(t) = \frac{a(1 - e^2)}{1 + e \cos\psi(t)}$$

and the angular velocity is

$$\dot{\psi}(t) = \frac{\sqrt{(m_1 + m_2)a(1 - e^2)}}{r(t)^2}$$

Treating the bodies as non-relativistic point masses (in the sense of question 7 of examples sheet 3), compute the corresponding energy-momentum tensor, the second moment of the energy distribution I_{ij}, and the metric perturbation $\bar{h}_{ij}$. (If you find the calculation too long then consider the simplified case of a circular orbit $e = 0$, or simplify further still by setting $e = 0$ and $m_1 = m_2$.) Show that the time average of the total power radiated in gravitational waves is

$$\langle P\rangle = \frac{32}{5}\frac{m_1^2 m_2^2 (m_1 + m_2)}{a^5} f(e), \qquad f(e) = \frac{1 + (73/24)e^2 + (37/96)e^4}{(1 - e^2)^{7/2}}$$

Note that $f(e)$ increases rapidly as $e \to 1$: $f(0.6) \sim 10$, $f(0.8) \sim 100$, $f(0.9) \sim 1000$. So highly eccentric orbits emit more gravitational radiation than circular ones. [This calculation is taken from P.C. Peters and J. Mathews, Physical Review 131, p435 (1963)]

2. As the above system emits gravitational radiation, it loses energy and angular momentum which causes the shape of the orbit to change gradually. We can model this by allowing a and e to be slowly varying functions of time. The energy of a Newtonian orbit is $E = -m_1 m_2/(2a)$ so a decreases over time. Setting $dE/dt = -\langle P\rangle$ gives an expression for da/dt. For a circular orbit ($e = 0$), use this to show that $a(t)$ reaches zero at a time

$$T = \frac{5a(0)^4}{256 m_1 m_2 (m_1 + m_2)}$$

Consider two black holes in a circular orbit, each with mass $30M_\odot$, which gives $m_1 = m_2 \approx 45$km. What is the time to merger if the initial distance between them is 1 astronomical unit (AU)? (1AU is the Earth-Sun distance: 1.5×10^8km) For comparison, the age of the Universe is about 14×10^9 years. How far apart are the black holes when the time to merger is 1 year? For comparison, the radius of the Sun is about 7×10^5km.

[The general case of an elliptical orbit with evolving a and e is considered in P.C. Peters, Physical Review 136, p1224 (1964). Emission of gravitional radiation causes e to decrease so the orbit becomes more circular over time.]

3. Let X be a p-form and Y and q-form on a manifold M. Show that the exterior derivative satisfies the properties $d(dX) = 0$, $d(X \wedge Y) = (dX) \wedge Y + (-1)^p X \wedge dY$ and $d(\phi^* X) = \phi^*(dX)$ where $\phi : N \to M$ for some manifold N.

4. A three-sphere can be parameterized by Euler angles (θ, ϕ, ψ) where $0 < \theta < \pi$, $0 < \phi < 2\pi$, $0 < \psi < 4\pi$. Define the following 1-forms

$$\sigma_1 = -\sin\psi\, d\theta + \cos\psi \sin\theta\, d\phi, \qquad \sigma_2 = \cos\psi\, d\theta + \sin\psi \sin\theta\, d\phi, \qquad \sigma_3 = d\psi + \cos\theta\, d\phi$$

Show that $d\sigma_1 = \sigma_2 \wedge \sigma_3$ with analogous results for $d\sigma_2$ and $d\sigma_3$.

5. If X is a p-form and V a vector field then define $i_V X$ to be the $(p-1)$-form that results from contracting V with the first index of X. Show that

$$\mathcal{L}_V X = i_V(dX) + d(i_V X)$$

6. Determine the connection 1-forms and curvature 2-forms for the Schwarzschild solution using the basis introduced in lectures. Hence determine the Riemann tensor components. Check that the Ricci tensor vanishes.

7. A spacetime has line element

$$ds^2 = -dt^2 + a(t)^2 dx^2 + b(t)^2 dy^2 + c(t)^2 dz^2$$

Calculate the connection 1-forms and the curvature 2-forms using the obvious orthonormal basis. Show that there is a non-trivial vacuum spacetime for which a, b, c are powers of t. This is the *Kasner solution* describing a homogeneous but anisotropic universe.

8. (Optional question.) In the notation of question 4, consider the following (Riemannian) metric on a three-sphere:

$$ds^2 = a^2 \sigma_1 \otimes \sigma_1 + b^2 \sigma_2 \otimes \sigma_2 + c^2 \sigma_3 \otimes \sigma_3$$

where a, b, c are positive constants. Calculate the Ricci tensor components in the orthonormal basis $e_1 = a\sigma_1$, $e_2 = b\sigma_2$, $e_3 = c\sigma_3$.

9. A spacetime has line element

$$ds^2 = f(\rho)^2 \left(-dt^2 + dz^2 + d\rho^2\right) + g(\rho)^2 d\phi^2$$

where (ρ, ϕ, z) have the same ranges as cylindrical polar coordinates: $\rho \geq 0$, $-\infty < z < \infty$ and ϕ is an angular coordinate: $\phi \sim \phi + 2\pi$.

(a) Determine the curvature 2-forms using the obvious orthonormal basis.

(b) Assume that this spacetime contains a time-independent magnetic field in the direction $\partial/\partial z$, described by the Maxwell 2-form

$$F = h(\rho) d\rho \wedge d\phi$$

(i) Show that the Maxwell equations reduces to an equation for $h(\rho)$. (ii) Determine the components of the associated energy-momentum tensor in the above basis.

(c) Show that the Einstein and Maxwell equations have the solution

$$f(\rho) = 1 + \frac{B^2}{4}\rho^2, \qquad g(\rho) = \rho f(\rho)^{-1}, \qquad h(\rho) = B\rho f(\rho)^{-2}$$

where B is a constant. This is the *Melvin magnetic universe*. It describes a magnetic flux tube held together by its gravitational self-attraction.

(d) Investigate the geometry of surfaces of constant t and z in the Melvin universe.

10. The *Reissner-Nordstrom* solution of the Einstein-Maxwell equations has metric

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2\left(d\theta^2 + \sin^2\theta \, d\phi^2\right) \qquad f(r) = 1 - \frac{2M}{r} + \frac{Q^2 + P^2}{r^2}$$

and Maxwell field

$$F = dA, \qquad A = -\frac{Q}{r}dt - P\cos\theta \, d\phi$$

where M, P, Q are constants. M can be interpreted as the total mass of this space-time. Assume that (t, r, θ, ϕ) is a right handed coordinate chart. Show that

$$\frac{1}{4\pi}\int_S \star F = Q, \qquad \frac{1}{4\pi}\int_S F = P$$

where S denotes a sphere at $r = \infty$ on a surface of constant t. (This proves that Q and P are the total electric and magnetic charges of this spacetime.)

11. The Lagrangian for the electromagnetic field is

$$L = -\frac{1}{16\pi} g^{ab} g^{cd} F_{ac} F_{bd}$$

where F_{ab} is written in terms of a potential A_a as $F = dA$. Show that this Lagrangian reproduces the Maxwell equations when A_a is varied, and that varying g_{ab} reproduces the energy-momentum tensor for the Maxwell field that was discussed in lectures.

12. (Optional question.) A test particle of rest mass m has a (timelike) worldline $x^\mu(\lambda)$, $0 \leq \lambda \leq 1$ and action

$$S = -m \int d\tau \equiv -m \int_0^1 \sqrt{-g_{\mu\nu}(x(\lambda))\dot{x}^\mu \dot{x}^\nu}\, d\lambda$$

where τ is proper time and a dot denotes a derivative with respect to λ.

(a) Show that varying this action with respect to $x^\mu(\lambda)$ leads to the geodesic equation.

(b) Show that the energy-momentum tensor of the particle in any chart is

$$T^{\mu\nu}(x) = \frac{m}{\sqrt{-g(x)}} \int d\tau u^\mu(\tau) u^\nu(\tau) \delta^4(x - x(\tau))$$

where u^μ is the 4-velocity of the particle.

(c) Conservation of the energy-momentum tensor is equivalent to the statement that

$$\int_R d^4x \sqrt{-g}\, v_\nu \nabla_\mu T^{\mu\nu} = 0$$

for any vector field v^μ and region R. By choosing v^μ to be compactly supported in a region intersecting the particle worldline, show that conservation of $T^{\mu\nu}$ implies that test particles move on geodesics. (This is an example of how the "geodesic postulate" of GR is a consequence of energy-momentum conservation.)

13. The action for the *Brans-Dicke* theory of gravity is given by

$$S = \frac{1}{16\pi} \int d^4x \sqrt{-g} \Big[R\phi - \frac{\omega}{\phi} g^{ab} \phi_{,a} \phi_{,b} + 16\pi L_{\text{matter}} \Big],$$

where ϕ is a scalar field and ω is a constant. Ordinary matter is included in the action L_{matter}. How is the Einstein equation modified, and what is the equation of motion for ϕ? (See Misner, Thorne and Wheeler or Carroll for further discussion of this theory.)

14. The bosonic fields of eleven-dimensional supergravity are the metric and a 4-form $G = dC$ where C is a 3-form potential. The action governing these fields is

$$S = \frac{1}{16\pi} \left[\int d^{11}x \sqrt{-g} \left(R - \frac{1}{48} G_{abcd} G^{abcd} \right) - \frac{1}{6} \int C \wedge G \wedge G \right]$$

(a) Show that this action is gauge invariant under $C \to C + d\Lambda$ where Λ is a 2-form (ignore surface terms).

(b) Vary the metric to determine the Einstein equation for this theory (note that the final term in the action does not depend on the metric).

(c) Vary C to obtain the equation of motion for the 4-form. Show that this can be written as

$$d \star G = \frac{1}{2} G \wedge G$$