Mathematical Tripos Part III Lent Term 2023
Black Holes: Examples Sheet 2 J.E. Santos

1. A general static, spherically symmetric metric can be written

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2 d\Omega^2,$$

where $d\Omega^2$ is the metric on a unit 2-sphere. Assume that $A(r)$ and $B(r)$ are analytic functions of r such that both have a simple zero at $r = r_+ > 0$ and are positive for $r > r_+$.

(a) Show that radial null geodesics are given by $t \pm r^* = \text{constant}$, where

$$r^* \equiv \int_{r_0}^{r} \frac{dx}{\sqrt{A(x)B(x)}},$$

with $r_0 > r_+$ an arbitrary constant. Show that $r^* \to -\infty$ as $r \to r_+$.

(b) Obtain the metric in ingoing Eddington-Finkelstein coordinates. Explain why this metric can be analytically continued through $r = r_+$.

2. Consider a particle with 4-velocity U in a stationary, asymptotically flat, space-time with timelike Killing vector field k. $E = -k \cdot U$ has the interpretation of "energy per unit mass measured at infinity" if the particle moves on a geodesic. For non-geodesic motion, this equation is used to *define* "energy per unit mass measured at infinity."

(a) Consider a unit mass particle P following an orbit of k at radius $r = r_P > 2M$ in the Schwarzschild spacetime. Assume that the force making this particle accelerate comes from a radial massless inelastic string, whose other end is held by an observer Q at infinity. If Q pulls the string through proper distance δS then what is the change δr_P in r_P ?

(b) What is the change δE in the energy of P measured by Q? This must equal the work $F\delta S$ done by Q where F is the force that the string exerts on Q, i.e., the tension at Q. Calculate F. Show that $F \to 1/(4M)$ as $r_P \to 2M$. What is the force measured by P as $r_P \to 2M$?

3. Use isotropic coordinates to prove that a surface of constant t in the Schwarzschild spacetime is an asymptotically flat end with $K_{ab} = 0$.

4. (a) Let (M, g) be the 2d Einstein static Universe with metric $ds^2 = -dt^2 + d\phi^2$ where $\phi \sim \phi + 2\pi$. Let S be the surface $t = 0$. Determine $D^+(S)$ and $J^+(S)$. (b) Do the same where (M, g) is now the spacetime obtained by deleting the point $t = \phi = 0$ from the Einstein static Universe. (c) Do the same for the Kruskal spacetime where S is the surface $t = 0$ in region I.

5. A perfect fluid has stress tensor $T_{ab} = (\rho + p)u_a u_b + p g_{ab}$, where ρ is the energy density, p the pressure, and u^a the 4-velocity of the fluid. Show that

(a) the dominant energy condition is obeyed if, and only if, $\rho \geq |p|$;

(b) the weak energy condition is obeyed if, and only if, $\rho \geq 0$ and $\rho + p \geq 0$;

(c) the null energy condition is obeyed if, and only if, $\rho + p \geq 0$;

(d) the strong energy condition is obeyed if, and only if, $\rho + 3p \geq 0$ and $\rho + p \geq 0$.

A cosmological constant has $p = -\rho$. Which energy conditions does it violate? (Consider both signs for ρ.)

6. Consider two Lorentzian metrics on a manifold M related by a conformal transformation $\bar{g} = \Omega^2 g$ where Ω is a positive function on M.

   (a) Show that g and $\bar{g}$ have the same null geodesics.

   (b) Show that the Ricci tensor of g is related to the Ricci tensor of $\bar{g}$ by

$$R_{ab} = \bar{R}_{ab} + 2\Omega^{-1}\bar{\nabla}_a\bar{\nabla}_b\Omega + \bar{g}_{ab}\bar{g}^{cd}\left(\Omega^{-1}\bar{\nabla}_c\bar{\nabla}_d\Omega - 3\Omega^{-2}\partial_c\Omega\partial_d\Omega\right)$$

   where $\bar{\nabla}$ is the Levi-Civita connection associated with $\bar{g}$.

   (c) Let ψ be a solution of the equation

$$g^{ab}\nabla_a\nabla_b\psi + \xi R\psi = 0$$

   We say that the equation is *conformally covariant* if there exists a constant p such that $\bar{\psi} \equiv \Omega^p\psi$ is a solution of the equation in a spacetime with metric $\bar{g} = \Omega^2 g$ whenever ψ solves the equation in a spacetime with metric g. Determine the value of ξ for which this equation is conformally covariant.

7. The Robinson-Bertotti metric is

$$ds^2 = -\lambda^2 dt^2 + M^2\left(\frac{d\lambda}{\lambda}\right)^2 + M^2 d\Omega^2$$

   This is the product $AdS_2 \times S^2$ where AdS_2 denotes 2d anti-de Sitter spacetime. By replacing the time coordinate t by one of the radial null coordinates $u = t + M/\lambda$, $v = t - M/\lambda$ show that the singularity at $\lambda = 0$ is merely a coordinate singularity. By introducing the new coordinates (U, V), defined by $u = \tan(U/2)$, $v = -\cot(V/2)$, obtain the maximal analytic extension of the RB metric and deduce its Penrose diagram (more precisely: deduce the Penrose diagram of the AdS_2 part of the RB metric). Is this spacetime globally hyperbolic?

8. Determine the Penrose diagram of *de Sitter spacetime* with metric

$$ds^2 = -dt^2 + H^{-2}\cosh^2(Ht)(d\chi^2 + \sin^2\chi d\Omega^2)$$

   where $H > 0$ is a constant and $0 \le \chi \le \pi$ ((χ, θ, ϕ) parameterize a round 3-sphere). (*Hint.* Use a coordinate transformation $t = t(\eta)$ to bring the metric to a form where it is manifestly conformal to the Einstein static Universe.)

9. Consider a vacuum spacetime that is asymptotically flat at null infinity. In lectures we introduced coordinates $(u, \Omega, \theta, \phi)$ such that $\mathcal{I}^+$ is $\Omega = 0$ and the "unphysical" metric satisfies

$$\bar{g}|_{\Omega=0} = 2dud\Omega + d\theta^2 + \sin^2 d\phi^2$$

   and, for small non-zero Ω, the corrections to this are $\mathcal{O}(\Omega)$ except for the uu, $u\theta$ and $u\phi$ components which are $\mathcal{O}(\Omega^2)$, and the $\Omega\Omega$ component which vanishes everywhere. The physical metric is $g = \Omega^{-2}\bar{g}$. Introduce a new coordinate $r = 1/\Omega$ and determine the form of the physical metric for large r, keeping track of the size of the subleading corrections. You should find that the uu component is $\mathcal{O}(1)$. Show that this component can be set to $-1 + \mathcal{O}(1/r)$ by a shift $r \to r + f(u, \theta, \phi)$. Finally define "asymptotically inertial" coordinates (t, x, y, z) by $t = u + r$ and (x, y, z) related to (r, θ, ϕ) as for spherical polars. Show that the spacetime metric becomes $-dt^2 + dx^2 + dy^2 + dz^2$ with corrections that are $\mathcal{O}(1/r)$ at large r.