**Mathematical Tripos Part III** Lent Term 2023
**Black Holes: Examples Sheet 3** J.E. Santos

1. Let $\Sigma$ be a spacelike hypersurface with future directed timelike unit normal $n^a$, induced metric $h_{ab} = g_{ab} + n_a n_b$ and extrinsic curvature $K_{ab} = h_a^c h_b^d \nabla_c n_d$. Let $S$ be a compact orientable 2d surface within $\Sigma$ with unit normal $m_a$. On $S$, let $U_\pm^a = (n^a \pm m^a)/\sqrt{2}$. (a) Show that $U_\pm^a$ are future-directed null vectors orthogonal to $S$ and $U_+ \cdot U_- = -1$. (b) Consider a null geodesic congruence containing the geodesics orthogonal to $S$ with tangent $U_\pm^a$ there. On $S$ we can choose (in the notation of lectures) $U^a = U_\pm^a$ and $N^a = U_\mp^a$. Show that the projection operator $P_b^a$ can be written as $P_b^a = h_b^a - m^a m_b$. (c) On $S$, the expansion of the geodesics orthogonal to $S$ is $\theta_\pm = P^{ab}\nabla_a U_b$. Since $P^{ab}$ is a projection onto directions tangential to $S$, this expression involves only derivatives tangential to $S$ so we can replace $U_b$ by its value on $S$, i.e., $U_{\pm b}$. Show that this gives

$$\theta_\pm = (h^{ab} - m^a m^b)K_{ab} \pm k$$

where $k$ is the trace of the extrinsic curvature of $S$ viewed as a surface in $\Sigma$. (d) Let $\Sigma$ be a *time-symmetric* hypersurface, i.e., $K_{ab} = 0$. Can $S$ be trapped? Show that $S$ is marginally trapped if, and only if, $k = 0$. (This is the condition for $S$ to be a *minimal surface* in $\Sigma$.) (e) Let $K_{ab} = J_{(a}M_{b)}$ where $J^a$ and $M^a$ are tangential to $\Sigma$ and orthogonal to each other. Assume that $M_a$ is tangent to $S$. Show that the results in (d) extend to this case. (A surface of constant $t$ in the Kerr geometry has $K_{ab}$ of this form.)

2. Consider the Reissner-Nordstrom solution with $M > e$ using advanced Eddington-Finkelstein coordinates. (a) Determine the Finkelstein diagram (i.e. show ingoing and outgoing radial null geodesics in a plot of $t_* = v - r$ against $r$). (b) Show that $r$ decreases along any causal curve in the region $r_- < r < r_+$.

3. (a) Prove that, if a vector field $\xi$ preserves the Maxwell field (i.e. $\mathcal{L}_\xi F = 0$) then locally there exists a scalar potential $\Phi$ such that $i_\xi F = d\Phi$. (*Hint:* Q2 of examples sheet 1.)

   (b) The equation of motion of a particle of charge $q$ and 4-velocity $U^a$ is $U^b\nabla_b U^a = (q/m)F^a{}_b U^b$. Let $\xi$ be a Killing vector field that preserves the Maxwell field. Show that $\xi \cdot U - (q/m)\Phi$ is conserved along the particle's worldline.

   (c) Deduce that, for a particle of mass $m$ moving in the equatorial plane ($\theta = \pi/2$) of a Reissner-Nordstrom black hole (with $Q > 0$, $P = 0$), the quantities $\mathcal{E} = (\Delta/r^2)dt/d\tau + qQ/(mr)$, and $h = r^2 d\phi/d\tau$ are constant ($\tau$ is proper time). Hence show that the radial motion is determined by the equation

$$\left(\frac{dr}{d\tau}\right)^2 + \frac{\Delta(r)}{r^2}\left(1 + \frac{h^2}{r^2}\right) = \left(\mathcal{E} - \frac{qQ}{mr}\right)^2.$$

   (d) What is the physical interpretation of the case $q/m = Q/M = 1$, $\mathcal{E} = 1$, $h = 0$?

   (e) *The Penrose process.* A particle $P_1$ falls from $r = \infty$ towards the black hole. Just before it crosses the event horizon, it decays into two other particles $P_2$ and $P_3$ where $P_2$ has charge $q < 0$. The decay happens such that $P_2$ initially has $dr/d\tau \approx 0$. $P_2$ subsequently falls into the black hole and $P_3$ escapes to $r = \infty$. Let $E_i \equiv m_i\mathcal{E}_i$ denote the energy of $P_i$ (which has mass $m_i$). Show that $E_1 > 0$ and $E_2 < 0$. Hence, by energy conservation, $E_3 > E_1$, i.e., the particle returning to infinity has more energy than the initial particle! This is consistent because $P_2$ has carried negative energy into the black hole. Hence energy (and charge) are extracted from the black hole in this process.

4. Obtain the Kerr-Newman solution in Kerr coordinates. When is there a regular event horizon? Show that the area of the event horizon of a Kerr-Newman black hole is $A = 8\pi(M^2 - e^2/2 + \sqrt{M^4 - e^2M^2 - J^2})$.

5. Let $E$ denote the maximum energy that can be extracted from a Kerr black hole in the Penrose process. The *efficiency* of this process is $\eta \equiv E/M$ where $M$ is the initial mass of the black hole. What is the largest possible value of $\eta$?

6. In the Kerr geometry, consider two spacelike surfaces $\Sigma$, $\Sigma'$ which both extend from $i^0$ to $\mathcal{H}^+$ with $\Sigma'$ lying entirely to the future of $\Sigma$. Let $H$ and $H'$ denote the intersections of $\Sigma$ and $\Sigma'$ with $\mathcal{H}^+$. Let $\mathcal{N}$ denote the portion of $\mathcal{H}^+$ from $H$ to $H'$. Let $J_a = -T_{ab}k^b$ be the conserved energy-momentum 4-vector.

   (a) Show that
   $$E(\Sigma') - E(\Sigma) = \int_{\mathcal{N}} \star J, \tag{1}$$
   where $E(\Sigma) \equiv -\int_\Sigma \star J$ is the total energy of matter fields on $\Sigma$, and similarly for $E(\Sigma')$. What is the physical interpretation of this formula?

   (b) Explain why the orientation of $\mathcal{N}$ used in this formula is given by $dv \wedge d\theta \wedge d\chi$ in Kerr coordinates (the orientation of spacetime is given by $dv \wedge dr \wedge d\theta \wedge d\chi$).

   (c) Show that $(\star J)_{v\theta\chi} = (r_+^2 + a^2)\sin\theta \xi^a J_a$.

   (d) Assume that matter obeys the dominant energy condition. Explain why $E(\Sigma') \leq E(\Sigma)$ for a Schwarzschild black hole (i.e. $a = 0$) but why this is not necessarily true for a Kerr black hole.

   (e) Now take the matter to be a massless real scalar field, with energy-momentum tensor $T_{ab} = \partial_a\Phi\partial_b\Phi - (1/2)g_{ab}(\partial\Phi)^2$. Consider a mode of this field with frequency $\omega$ and azimuthal quantum number $\nu$, i.e., $\Phi = \mathrm{Re}[\Phi_0(r,\theta)\exp(-i\omega v + i\nu\chi)]$. Show that the RHS of equation (1) is positive for $0 < \omega < \nu\Omega_H$. (Note that $\xi \cdot k = 0$ on $\mathcal{H}^+$ because $\mathcal{H}^+$ must be invariant under an isometry hence any Killing field must be tangent to $\mathcal{H}^+$.)

   This example shows that energy can be extracted from a black hole by scattering waves off it. This is called *superradiant scattering*.

7. (a) Calculate the ADM mass of the Reissner-Nordstrom solution.

   (b) Calculate the electric and magnetic charges, the Komar mass, and the Komar angular momentum of the Kerr-Newman solution.

8. (a) Let $(M, g)$ be a stationary vacuum spacetime containing an hypersurface $\Sigma$ such that the initial data induced on $\Sigma$ is geodesically complete and asymptotically flat with 1 end. Prove that the Komar mass must vanish and hence, by the positive energy theorem, that the spacetime must be flat. (This is a version of *Lichnerowicz's theorem* which excludes the existence of gravitational solitons, i.e., stationary configurations of the gravitational field that are not black holes.)

9. Four-dimensional anti-de Sitter space-time ($\mathrm{AdS}_4$) with radius of curvature $\ell$ has metric
   $$ds^2 = -U(r)dt^2 + \frac{dr^2}{U(r)} + r^2 d\Omega^2,$$
   where $U(r) = 1 + r^2/\ell^2$.

   (a) Show that, along a null geodesic with affine parameter $\lambda$, $r \to \infty$ and $t \to$ constant as $\lambda \to \pm\infty$.

(b) Construct the conformal compactification of $\mathrm{AdS}_4$ by defining a new radial coordinate $\chi$ by $r = \ell \tan \chi$.

(c) Is $\mathrm{AdS}_4$ globally hyperbolic?

(d) The Schwarzschild-AdS metric, given by setting $U(r) = 1 - 2M/r + r^2/\ell^2$ above, is the unique spherically symmetric solution of the vacuum Einstein equation with a negative cosmological constant. Let $M(r)$ and $\bar{M}(r)$ denote the Komar mass associated with a sphere of constant $r$ and $t$ in the Schwarzschild-AdS and AdS metrics respectively. Show that $M$ and $\bar{M}$ both diverge as $r \to \infty$ but $M(r) - \bar{M}(r)$ has a finite limit. (Note that $r$ is invariantly defined in a spherically symmetric spacetime, so this prescription for calculating the mass is coordinate-independent.)

(e) Consider Schwarzschild-adS with $M > 0$. Show that there is a Killing horizon of $\partial/\partial t$ at $r = r_+ > 0$ where $U(r_+) = 0$. Plot the surface gravity $\kappa$ as a function of $M$. How does this differ from the corresponding plot for a Schwarzschild black hole?

10. ($\star\star$) Show that the Kerr-Newman geometry has a curvature singularity at $(r, \theta) = (0, \pi/2)$. Furthermore, show that this singularity has a ring like structure.