Mathematical Tripos Part III **Prof G W Gibbons**
Lent Term 2006

Applications of Differential Geometry to Physics
Example Sheet 1

1. Check that the statement :

$$\omega_{ab\ldots c} = \omega_{[ab\ldots c]}$$

about the components of a p-form ω is independent of the basis chosen for V.

If$\{\omega^a\}$ is a basis for V^*, show that the set of all possible combinations of

$$\omega^{a_1} \wedge \omega^{a_2} \ldots \omega^{a_p}, a_1 < a_2 \ldots < a_p$$

gives a basis for $\Lambda^p(V)$. Hence show that if $\omega \wedge \alpha = 0$ for a one-form ω and p-form α then

$$\alpha = \omega \wedge \beta$$

for some $(p-1)$-form β.

2. Show that $\omega \wedge \nu(X, Y) = 0$ for all vectors X and Y contained in the intersection of the hyperplanes through origin of V associated to the one-forms ω and ν.

Show how a two-form α provides a linear map from V to V^* whose matrix elements are the components α_{ab} of α.

A non-zero two-form α is said to be *simple* if $\alpha = \omega \wedge \nu$ for one forms ω and ν. Show that α is simple if and only if the rank of the matrix is two. Identify the set of vectors in V belonging to the kernel of α. Show that the Maxwell two form F is simple if and only if $\mathbf{E}.\mathbf{B} = 0$. Calculate $F \wedge F$ and comment on your result.

3. Given a metric g on V we can look for eigen-vectors X and eigenvalues λ of a two-form α with respect to g, i.e. vectors $X \in V$ such that

$$\alpha(X,) = \lambda g(X,)$$

Show that if g is positive definite then necessarily the eigenvalues must vanish but if g has indefinite signature then λ need not vanish but in that case X must be a null vector.

4. Suppose that $n = \dim V = 2m$ is even. Introduce a positive definite metric g and use it to identify V and V^* in the usual way so that the components, g_{ab}, of g may be taken to be δ_{ab}. From now on we suspend the Einstein summation convention. Show, by diagonalizing the matrix $\sum_c \alpha_{ac}\alpha_{cb}$ with respect to g_{ab}, that a basis $\{\mu^i, \nu^i\}$ for V^* can be found such

$$\alpha(\mu^i, \,) = \lambda_i \nu^i$$

$$\alpha(\nu^i, \ ) = -\lambda_i \mu^i$$

and thus that

$$\alpha = \sum_{i=1}^{i=m} \lambda_i \mu^i \wedge \nu^i$$

Use this result to relate the m-fold wedge product $\alpha \wedge \alpha \ldots \wedge \alpha$ to $\det(\alpha_{ab})$. Deduce that the rank of the two form must be even. What happens if $\dim V$ is odd?

5. Suppose that F is an odd form which is also self-dual. Show that

$$(F, F) = 0\,.$$

Comment on the case when F is a one-form.

6.Show, using differential forms, that

$$F = Q\frac{dt \wedge dr}{r^2} + P \sin\theta d\theta \wedge d\phi$$

satisfies all of Maxwell's equations in the spacetime metric

$$ds^2 = -\Delta(r)dt^2 + \frac{dr^2}{\Delta(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2).$$

for any choice of the function $\Delta(r)$ and constants Q and P. Find a suitable vector potential, valid locally. Show that if $P \neq 0$ there can be no globally defined vector potential. Calculate the action of the Maxwell field in a spacetime volume given by $r_1 < r < r_2, t_1 < t < t_2, 0 < \phi < 2\pi, 0 < \theta < \pi$.

7. Justify the expression $d{*}d\Phi = 0$ for the massless wave equation. Write out this equation in the metric used in the previous question.

8. Show, both directly, and starting from the action principle, that

$$\nabla_\alpha F^{\alpha\beta\ldots\gamma} = \frac{1}{\sqrt{-g}}\partial_\alpha\big(\sqrt{-g}F^{\alpha\beta\ldots\gamma}\big)\,,$$

for a p-form F. Show further, that if F is a middle-dimensional form, i.e. if $n = 2p$, then the systen of equations $dF = 0$ and $d \star F = 0$, is conformally invariant in the sense that if F, g is a solution then so is $F, \Omega^2 g$, where g is the metric. Calculate, using the formula

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}\,,$$

the energy momentum tensor of the system, where S is the action functional and $\frac{\delta S}{\delta g^{\mu\nu}}$ is the functional derivative. Show in particular that for a middle dimensional form, the energy momentum tensor is trace free How are these facts related to the conformal invariance of the action S?

9. The Hodge-DeRham Laplacian Δ is defined by

$$\Delta = \delta d + d\delta.$$

Show that Δ commutes with d and δ. Use the Ricci identity to show that on one-forms:

$$\Delta A_\mu = -\nabla^2 A_\mu + R_\mu^\lambda A_\lambda,$$

where $R_{\mu\nu}$ is the Ricci tensor. Write down the equation satisfied by the vector potential of a solution of Maxwell's equations in Lorentz gauge, $\nabla^\mu A_\mu = 0$.

10 Let A be a p-form and F a $(p+1)$-form. Expand out the equation

$$(d - \delta)(F + mA) = m(F + mA).$$

Show that if $p = 0$ then these equations are equivalent to the massive wave equation for A, i.e.

$$-\nabla_\mu \nabla^\mu A + m^2 A = 0$$

If $p = 1$ one obtains the Proca equations for a massive spin one particle. Write these down in components and give in particular the corresponding wave equation for A_μ.

11.One might think that the existence of a scalar-density constructed from the Maxwell two-form F given by $\sqrt{\det F_{\alpha\beta}}$ would allow one to construct a Lagrangian on any manifold independently of the existence of any metric. Explain the fallacy in this argument by establishing that, locally at least,

$$F \wedge F = dG$$

where G is a three-form constructed from F and A whose precise form should be given.

12. A certain field theory is based on maps $\phi : \mathcal{M} \to S^3$ from 4-dimensional spacetime $\mathcal{M}$ *to* a unit 3-sphere equipped with its standard metric. The map ϕ is given by

$$x \to \phi^a(x), a = 1, 2, 3, 4$$

subject to the constraint:

$$r^2 \equiv \phi^a \phi^a = 1$$

Given that the volume form on $\mathbb{R}^4$ satisfies

$$d\phi^1 \wedge d\phi^2 \wedge d\phi^3 \wedge d\phi^4 = dr \wedge r^3 \eta$$

where η is the volume form on the unit 3-sphere, show that

$$\eta = \frac{1}{3!}\phi^a \epsilon_{abcd} d\phi^b \wedge d\phi^c \wedge d\phi^d.$$

Let $\alpha = \phi^* \eta$ be the pull-back of η to the spacetime manifold $\mathcal{M}$ and let $*_\mathcal{M}$ denote the Hodge star operator with respect to the spacetime metric $g_{\alpha\beta}$. Show, using the behaviour of d under pullback, that

$$j = *_\mathcal{M} \alpha$$

defines a current one-form j on spacetime which is conserved, i.e. $\delta j = -\nabla^\mu j_\mu = 0$ independently of any field equations that ϕ^a may satisfy. Use Stokes's theorem to show that

$$N = \frac{1}{2\pi^2} \int_\Sigma \alpha$$

is independent of the spacelike hypersurface Σ over which it is integrated, as long as boundary terms at infinity may be neglected. Explain why (given that the volume of $S^3 = 2\pi^2$) that N will take integer values if suitable conditions on ϕ^a hold near infinity. Show, in the case that Σ is a $t =$ constant surface in Minkowski spacetime that

$$N = \frac{1}{2\pi^2} \int d^3x \frac{1}{3!} \phi^a \nabla \phi^b \times \nabla \phi^c . \nabla \phi^d \epsilon_{abcd}.$$

13. (**Optional Extra**)

The Clifford algebra $\mathrm{Cliff}(s,t;\mathbb{R})$ is the associative algebra over $\mathbb{R}$ generated by the relations

$$\gamma_a \bullet \gamma_b + \gamma_b \bullet \gamma_a = 2g_{ab} \,,$$

where γ_a is a basis for an n -dimensional vector space with, not necessarily invertable metric g_{ab}, with s positive and t negative eigen values. Evidently

$$\mathrm{Cliff}(0,0;\mathbb{R}) \equiv \Lambda^\star(\mathbb{R}^n) \,.$$

In what follows, assume that g_{ab} is non-degenerate and denote a vector space carrying such a metric by by $\mathbb{E}^{s,t}$ with $s + t = n$.

Show that $\mathrm{Cliff}(0,1;\mathbb{R}) \equiv \mathbb{C}$ and $\mathrm{Cliff}(0,2;\mathbb{R}) \equiv \mathbb{H}$ where $\mathbb{H}$ denotes the quarternions. If σ_i are Pauli matrices, show, by picking $\gamma_i = \sigma_i$ that $\mathrm{Cliff}(0,1 : \mathbb{R}) \equiv M_2(\mathbb{C})$, where $M_n(\mathbb{C})$ is the algebra of $n \times n$ matrices with complex coeficients.

A representation of a Clifford algebra may be given by a set of so-called 'Dirac gamma matrices 'acting on some vector space satisfying the Clifford relations.

Suppose that a_i and $a_i^\dagger$ are as defined in the lectures associated to $\Lambda^\star(\mathbb{E}^m)$.

By taking

$$\gamma_i = a_i + a_i^\dagger \,, \qquad i = 1, 2, \ldots, m \,.$$

and

$$\gamma_{m+i} = a_i - a_i^\dagger \,, \qquad i = 1, 2, \ldots, m \,.$$

show that there is a real representation of $\mathrm{Cliff}(m,m;\mathbb{R})$ on $\Lambda^\star(\mathbb{E}^m)$. What is the dimension of the matrices?