A1c **Vectors and Matrices: Example Sheet 3** Michaelmas 2017

A $*$ denotes a question, or part of a question, that should not be done at the expense of questions on later sheets. Starred questions are **not** *necessarily harder than unstarred questions.*

Corrections and suggestions should be emailed to `N.Peake@damtp.cam.ac.uk`.

1. Given that A is the real matrix
$$\begin{pmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{pmatrix},$$
show with the aid of row operations that
$$\det \mathsf{A} = (a-b)(b-c)(c-a)(ab+bc+ca).$$
[*Recall that the value of the determinant is unchanged if a linear combination of any two rows is added to the third row.*]

2. Show that
$$\begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} = x^3+y^3+z^3-3xyz \equiv \Delta.$$
Show, by row operations, that
$$x+y+z, \quad x+\omega y+\omega^2 z, \quad x+\omega^2 y+\omega z$$
are factors of Δ, where ω is a complex cube root of unity. Show, by considering the coefficients of x^3, that Δ is equal to the product of the three indicated factors.

3. If A is a $(2n+1)\times(2n+1)$ antisymmetric matrix ($n \in \mathbb{N}$), calculate $\det \mathsf{A}$.

4. Let D be the $n \times n$ matrix which has the entry p, $p \neq 1$, at each place on the main diagonal and unity in every other position. Show that $\det \mathsf{D} = (p+n-1)(p-1)^{n-1}$.

5. Identify the cofactors Δ_{ij} of a_{ij} in the matrix
$$\mathsf{A} = \{a_{ij}\} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & 2 \end{pmatrix}.$$
Verify the identity $a_{ij}\Delta_{ik} = \delta_{jk} \det \mathsf{A}$, and hence construct the matrix A^{-1}.
Use your result to solve the equations
$$\begin{aligned} x+y+z &= 1\,, \\ x+2y+3z &= -5\,, \\ 3x-2y+2z &= 4\,. \end{aligned}$$
Verify that your answers for (x, y, z) do indeed satisfy the equations.

6. For each real value of t, determine whether or not there exist solutions to the simultaneous equations
$$\begin{aligned} x+y+z &= t \\ tx+2z &= 3 \\ 3x+ty+5z &= 7, \end{aligned}$$
exhibiting the most general form of such solutions when they exist.

∗7. Let A be a real 3×3 matrix, and let $\mathbf{d}$ be a 3 component column vector. Explain briefly how the general solution of the matrix equation $\mathsf{A}\mathbf{x} = \mathbf{d}$, where $\mathbf{x}$ is a 3 component column vector, depends on the kernel and image of the linear map $\mathbf{x} \mapsto \mathsf{A}\mathbf{x}$.

Consider the case

$$\mathsf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & a^2 & b^2 \end{pmatrix}, \qquad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad \mathbf{d} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Find the kernel and image of the corresponding map, noting the different possibilities according to different values of a and b.

For which values of a and b do the equations $\mathsf{A}\mathbf{x} = \mathbf{d}$ have (i) a unique solution, (ii) more than one solution, (iii) no solution? For each pair (a, b) satisfying (ii), give the solutions as the sum of a fixed solution and the general solution of the corresponding homogeneous equations.

8. Find the eigenvalues and eigenvectors of the matrix

$$\mathsf{A} = \begin{pmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where neither of the complex constants α and β vanishes. Find the conditions for which (a) the eigenvalues are real, and (b) the eigenvectors are orthogonal. Hence show that both conditions are jointly satisfied if and only if A is Hermitian.

Recall both that the scalar product for two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{C}^3$ *is defined as*

$$\mathbf{u} \cdot \mathbf{v} = u_1^* v_1 + u_2^* v_2 + u_3^* v_3\,,$$

where * *denotes a complex conjugate, and that* $\mathbf{u}$ *and* $\mathbf{v}$ *are said to be orthogonal if* $\mathbf{u} \cdot \mathbf{v} = 0$.

9. (a) Find a 3×3 real matrix with eigenvalues 1, i, $-i$. *Hint:* think geometrically.
   (b) Construct a 3×3 non-zero real matrix which has all three eigenvalues zero.

10. (a) Let A be a square matrix such that $\mathsf{A}^m = 0$ for some integer m. Show that every eigenvalue of A is zero.
    (b) Let A be a real 2×2 matrix which has non-zero non-real eigenvalues. Show that the non-diagonal elements of A are non-zero, but that the diagonal elements may be zero.

11. Let Q be a $(2n + 1) \times (2n + 1)$ orthogonal matrix ($n \in \mathbb{N}$) with $\det \mathsf{Q} = 1$. Show that Q has a unit eigenvalue. Give a geometric interpretation of your result for 3×3 matrices.

∗12. Suppose that A is an $n \times n$ square matrix and that A^{-1} exists. Show that if A has characteristic equation $a_0 + a_1 t + \cdots + a_n t^n = 0$, then A^{-1} has characteristic equation

$$(-1)^n \det(\mathsf{A}^{-1})(a_n + a_{n-1} t + \cdots + a_0 t^n) = 0.$$

Hints. Take $n = 3$ in this question if you wish, but treat the general case if you can. It should be clear that λ is an eigenvalue of A if and only if $1/\lambda$ is an eigenvalue of A^{-1}, but this result says more than this (about multiplicities of eigenvalues). You should use properties of the determinant to solve this problem, for example, $\det(\mathsf{A})\det(\mathsf{B}) = \det(\mathsf{AB})$. You should also state explicitly why we do not need to worry about zero eigenvalues.

13. For each of the three matrices below,
    (a) compute their eigenvalues (as often happens in exercises and seldom in real life each eigenvalue is a small integer);
    (b) for each real eigenvalue λ compute the dimension of the eigenspace $\{\mathbf{x} \in \mathbb{R}^3 : \mathsf{A}\mathbf{x} = \lambda\mathbf{x}\}$;
    (c) determine whether or not the matrix is diagonalizable as a map of $\mathbb{R}^3$ into itself.

$$\begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix}, \qquad \begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}, \qquad \begin{pmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix}.$$