Mathematical Tripos Part IA **Michaelmas Term 2023**
Vectors and Matrices **Professor N. Peake**

Example Sheet 3

1. Let A and B be $n \times n$ hermitian matrices and U be an $n \times n$ unitary matrix. Show that
(i) $\mathrm{tr}A$ and $\det A$ are real; (ii) $AB + BA$ is hermitian; (iii) $i(AB - BA)$ is hermitian and has zero trace; (iv) $\mathrm{tr}(AB)$ and $\det(AB)$ are real; (v) $\mathrm{tr}(UAU^\dagger) = \mathrm{tr}(A)$ and $\det(UAU^\dagger) = \det(A)$.

2. For the real matrix

$$M = \begin{pmatrix} a & a^2 & bc \\ b & b^2 & ca \\ c & c^2 & ab \end{pmatrix},$$

show with the aid of row operations that

$$\det M \;=\; (a-b)(b-c)(c-a)(ab+bc+ca)\,.$$

[*Recall*: the value of a determinant is unchanged if a multiple of one row is added to another row.]

3. Show by direct evaluation of the determinant that

$$\Delta(x,y,z) \;\equiv\; \begin{vmatrix} x & y & z \\ z & x & y \\ y & z & x \end{vmatrix} \;=\; x^3+y^3+z^3-3xyz\,.$$

Now show by row or column operations that

$$x+y+z\,, \quad x+\omega y+\omega^2 z\,, \quad x+\omega^2 y+\omega z$$

are factors of $\Delta(x,y,z)$, where ω is a complex cube root of unity. By considering the coefficients of x^3, deduce that $\Delta(x,y,z)$ is the product of the three factors above.

4. If A is a $(2n+1) \times (2n+1)$ antisymmetric matrix, find $\det A$.

5. Let D be the $n \times n$ matrix $(n > 1)$ which has entries p at each place on the main diagonal and entries 1 in every other position. Show that $\det D = (p+n-1)(p-1)^{n-1}$.

6. Calculate the cofactors Δ_{ij} for the matrix

$$A \;=\; \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 3 & -2 & 2 \end{pmatrix}.$$

Convert the suffix notation equation $A_{ij}\Delta_{ik} \;=\; \delta_{jk} \det A$ into matrix notation and check that your calculated cofactors do indeed satisfy that equation. Find A^{-1}.

Use your result to solve the equations

$$\begin{aligned} x+y+z &= 1 \\ x+2y+3z &= -5 \\ 3x-2y+2z &= 4\,. \end{aligned}$$

Verify that your answers for (x,y,z) do indeed satisfy the equations.

7. For each real value of t, determine whether or not there exist solutions to the simultaneous equations

$$\begin{aligned} x+y+z &= t \\ tx+2z &= 3 \\ 3x+ty+5z &= 7 \end{aligned}$$

exhibiting the most general form of such solutions when they exist.

Copyright © 2023 University of Cambridge. Not to be quoted or reproduced without permission.

8. Let M be a real 3×3 matrix, and let $\mathbf{d}$ be a 3 component column vector. Explain briefly how the general solution of the matrix equation $M\mathbf{x} = \mathbf{d}$, where $\mathbf{x}$ is a 3 component column vector, depends on the image and kernel of the linear map $\mathbf{x} \mapsto M\mathbf{x}$.

Consider the case

$$M = \begin{pmatrix} 1 & 1 & 1 \\ 1 & a & b \\ 1 & a^2 & b^2 \end{pmatrix}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \quad \mathbf{d} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Find the image and kernel of the corresponding map, noting the different possibilities according to different values of a and b.

For which values of a and b do the equations $M\mathbf{x} = \mathbf{d}$ have (i) a unique solution, (ii) more than one solution, (iii) no solution? For each pair (a, b) satisfying (ii), give the solutions as the sum of a fixed solution and the general solution of the corresponding homogeneous equations.

9. (a) Find a 3×3 real matrix with eigenvalues 1, i, $-i$. *Hint:* think geometrically.

(b) Construct a 3×3 non-zero real matrix which has all three eigenvalues zero.

(c) Let A be a square matrix such that $A^m = 0$ for some positive integer m. Show that every eigenvalue of A is zero.

(d) Let M be a real 2×2 matrix which has non-real eigenvalues. Show that the non-diagonal elements of M are non-zero, but that the diagonal elements may be zero.

10. Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where α, β are non-zero complex numbers. Find the conditions for which (i) the eigenvalues are real, and (ii) eigenvectors for different eigenvalues are orthogonal. Show that both these conditions hold if and only if A is hermitian. [*Recall*: the complex inner product of vectors $\mathbf{z}$, $\mathbf{w}$ in $\mathbb{C}^3$ is given by

$$\mathbf{z}^\dagger \mathbf{w} = \bar{z}_1 w_1 + \bar{z}_2 w_2 + \bar{z}_3 w_3$$

and $\mathbf{z}$ and $\mathbf{w}$ are said to be orthogonal if $\mathbf{z}^\dagger \mathbf{w} = 0$.]

11. Let Q be a $(2n+1) \times (2n+1)$ orthogonal matrix with $\det Q = 1$. Show that 1 is an eigenvalue of Q and give a geometric interpretation of this result when Q is a 3×3 matrix.

What can be said if $\det Q = -1$?

12. For each of the matrices

$$A = \begin{pmatrix} 5 & -3 & 2 \\ 6 & -4 & 4 \\ 4 & -4 & 5 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix}, \quad C = \begin{pmatrix} 7 & -12 & 6 \\ 10 & -19 & 10 \\ 12 & -24 & 13 \end{pmatrix},$$

(i) compute their eigenvalues (as often happens in exercises, each eigenvalue is a small integer);

(ii) for each eigenvalue λ compute the dimension of the eigenspace $\{\mathbf{x} \in \mathbb{R}^3 : M\mathbf{x} = \lambda\mathbf{x}\}$;

(iii) hence determine whether or not the matrix is diagonalizable.

13. Suppose that A is an $n \times n$ matrix and that A^{-1} exists. Show that if A has characteristic polynomial $\chi_A(t) = a_0 + a_1 t + \ldots + a_n t^n$, then A^{-1} has characteristic polynomial

$$\chi_{A^{-1}}(t) = (-1)^n \det(A^{-1}) \, (a_n + a_{n-1} t + \ldots + a_0 t^n).$$

Hint: Use properties of the determinant such as $\det(A)\det(B) = \det(AB)$.

Comments to: np100@cam.ac.uk

Copyright © 2023 University of Cambridge. Not to be quoted or reproduced without permission.