A1d **Vectors and Matrices: Example Sheet 4** Michaelmas 2017

A $*$ denotes a question, or part of a question, that should not be done at the expense of questions later on the sheet. Starred questions are **not** *necessarily harder than unstarred questions.*

Corrections and suggestions should be emailed to `N.Peake@damtp.cam.ac.uk`.

1. A matrix A is said to be *upper triangular* if $A_{ij} = 0$ if $i > j$, i.e. if

$$\mathsf{A} = \begin{pmatrix} A_{11} & A_{12} & \dots & A_{1\,n} \\ 0 & A_{22} & \ddots & A_{2\,n} \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & A_{nn} \end{pmatrix}.$$

Show that the eigenvalues are $\lambda_i = A_{ii}$ ($i = 1, \dots, n$, and obviously no sum).

2. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_m\}$ and $\{\mathbf{f}_1, \dots, \mathbf{f}_n\}$ be bases for $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively, and let $\mathcal{A}$ be a linear mapping from $\mathbb{R}^m$ to $\mathbb{R}^n$. Explain how to represent $\mathcal{A}$ by a matrix relative to the given bases.

(a) Taking $m = 2$, $n = 3$ and $\mathcal{A}$ as the linear mapping for which

$$\mathcal{A}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}2\\1\\5\end{pmatrix}, \quad \mathcal{A}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}7\\0\\3\end{pmatrix},$$

where components are with respect to the standard bases for $\mathbb{R}^2$ and $\mathbb{R}^3$, find the matrix of $\mathcal{A}$ with respect to the bases

$$\mathbf{e}_1 = \begin{pmatrix}1\\-1\end{pmatrix}, \quad \mathbf{e}_2 = \begin{pmatrix}-3\\2\end{pmatrix}; \qquad \mathbf{f}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \mathbf{f}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \mathbf{f}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

(b) The mapping $\mathcal{A}$ of $\mathbb{R}^3$ to itself is a reflection in the plane $x_1 \sin\theta = x_2 \cos\theta$. Find the matrix A of $\mathcal{A}$ with respect to any basis of your choice, which should be specified.

3. The linear map $\mathcal{A} : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by

$$\begin{pmatrix}x\\y\end{pmatrix} \longmapsto \begin{pmatrix}x'\\y'\end{pmatrix} = \mathsf{A}\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}-5x+9y\\-4x+7y\end{pmatrix}.$$

Find the matrix B of the map $\mathcal{A}$ relative to the basis

$$\left\{\begin{pmatrix}3\\2\end{pmatrix}, \begin{pmatrix}1\\1\end{pmatrix}\right\},$$

and interpret the map geometrically. Hence show that, for each positive integer n,

$$\mathsf{B}^n - \mathsf{I} = n(\mathsf{B} - \mathsf{I}),$$

where $\mathsf{I} = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$. Hence evaluate A^n. Verify that $\det(\mathsf{A}^n) = (\det \mathsf{A})^n$.

$*$4. Show that similar matrices have the same rank.

5. Show that the matrix

$$\mathsf{A} = \begin{pmatrix} 0 & 1 & 0\\ -4 & 4 & 0\\ -2 & 1 & 2\end{pmatrix}$$

has characteristic equation $(t-2)^3 = 0$. Explain (without doing any further calculations) why A is not diagonalisable.

6. (a) Find a, b and c such that the matrix

$$\begin{pmatrix} 1/3 & 0 & a \\ 2/3 & 1/\sqrt{2} & b \\ 2/3 & -1/\sqrt{2} & c \end{pmatrix}$$

is orthogonal. Does this condition determine a, b and c uniquely?

(b) Let

$$\mathsf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } \mathsf{P} = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

Do you expect PAP^{-1} to be symmetric? Compute PAP^{-1}. Were you right?

*7. (a) Show that the Cayley-Hamilton theorem is true for all 2×2 matrices.

(b) Let

$$\mathsf{A} = \begin{pmatrix} 3 & 4 \\ -1 & -1 \end{pmatrix}.$$

Find the characteristic equation for A. Verify that $\mathsf{A}^2 = 2\mathsf{A} - \mathsf{I}$. Is A diagonalisable ?

Show by induction that A^n lies in the two-dimensional subspace (of the space of 2×2 real matrices) spanned by A and I, i.e. show that there exists real numbers α_n and β_n such that

$$\mathsf{A}^n = \alpha_n \mathsf{A} + \beta_n \mathsf{I}\,.$$

Find a recurrence relation (i.e. a difference equation) for α_n and β_n, and hence find an explicit formula for A^n.

8. Determine the eigenvalues and eigenvectors of the symmetric matrix

$$\mathsf{A} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 2 & 0 \\ 1 & 0 & 2 \end{pmatrix}.$$

Use an identity of the form $\mathsf{P}^{\mathrm{T}}\mathsf{AP} = \mathsf{D}$, where D is a diagonal matrix, to find A^{-1}.

*9. Show that the eigenvalues of a unitary matrix have unit modulus. Show that if a unitary matrix has distinct eigenvalues then the eigenvectors are orthogonal.

10. A skew-Hermitian matrix, W, is one such that $\mathsf{W}^\dagger = -\mathsf{W}$. What can be said about the eigenvalues of a skew-Hermitian matrix? (*Hint: consider* $\mathsf{H} = i\mathsf{W}$)?

If S is real symmetric and T is real skew-symmetric, show that $\mathsf{T} \pm i\mathsf{S}$ is skew-Hermitian. State a property of the eigenvalues of $\mathsf{T} + i\mathsf{S}$ and hence, or otherwise, show that

$$\det(\mathsf{T} + i\mathsf{S} - \mathsf{I}) \neq 0\,.$$

Show that the matrix

$$\mathsf{U} = (\mathsf{I} + \mathsf{T} + i\mathsf{S})(\mathsf{I} - \mathsf{T} - i\mathsf{S})^{-1}$$

is unitary. For

$$\mathsf{S} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \qquad \mathsf{T} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

show that the eigenvalues of U are $\pm(1 - i)/\sqrt{2}$.

*11. *This is a continuation of question 8 on Example Sheet 2.*

As in question 8 on Example Sheet 2 consider the linear map $\mathcal{S} : \mathbb{R}^2 \to \mathbb{R}^2$

$$\mathbf{x} \mapsto \mathbf{x}' = \mathbf{x} + \lambda(\mathbf{b} \cdot \mathbf{x})\,\mathbf{a} \qquad (*)$$

where λ is a real scalar, $\mathbf{a}$ and $\mathbf{b}$ are fixed orthogonal unit vectors. Let $\mathsf{S}(\lambda, \mathbf{a}, \mathbf{b})$ be the matrix with elements S_{ij} such that $x'_i = S_{ij}x_j$. Give diagrams illustrating the shears

$$\mathsf{S}_1 = \mathsf{S}(\lambda, \mathbf{i}, \mathbf{j})\,, \quad \mathsf{S}_2 = \mathsf{S}(\lambda, \mathbf{j}, -\mathbf{i}).$$

Show that S_1 and S_2 are related by a similarity transformation

$$\mathsf{S}_2 = \mathsf{R}^{-1}\mathsf{S}_1\mathsf{R}\,, \quad \mathsf{R} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Now let $\mathcal{S}$ be the map defined by $(*)$ but from $\mathbb{R}^3$ to $\mathbb{R}^3$, and let $\mathbf{i}, \mathbf{j}, \mathbf{k}$ be unit vectors along the three perpendicular axes. Find the matrix S in each of the cases

$$\text{(i)} \quad \mathbf{a} = \mathbf{i}\,,\ \mathbf{b} = \mathbf{j}\,, \qquad \text{(ii)} \quad \mathbf{a} = \mathbf{j}\,,\ \mathbf{b} = \mathbf{k}\,, \qquad \text{(iii)} \quad \mathbf{a} = \mathbf{k}\,,\ \mathbf{b} = \mathbf{i}\,,$$

and interpret the corresponding simple shears. Show that any matrix of the form

$$\begin{pmatrix} 1 & \lambda & \mu \\ 0 & 1 & \nu \\ 0 & 0 & 1 \end{pmatrix}$$

can be displayed (not necessarily uniquely) as the product of matrices of simple shears.

*12. Diagonalize the quadratic form

$$\mathcal{F} = (a\cos^2\theta + b\sin^2\theta)x^2 + 2(a-b)(\sin\theta\cos\theta)xy + (a\sin^2\theta + b\cos^2\theta)y^2,$$

and identify the principal axes.

13. Find all eigenvalues, and an orthonormal set of eigenvectors, of the matrices

$$\mathsf{A} = \begin{pmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{pmatrix} \quad \text{and} \quad \mathsf{B} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2. \end{pmatrix}$$

Hence sketch the surfaces

$$5x^2 + 3y^2 + 3z^2 + 2\sqrt{3}xz = 1 \quad \text{and} \quad x^2 + y^2 + z^2 - xy - yz - zx = 1\,.$$

14. Let Σ be the surface in $\mathbb{R}^3$ given by

$$2x^2 + 2xy + 4yz + z^2 = 1.$$

By writing this equation as

$$\mathbf{x}^{\mathrm{T}}\mathsf{A}\mathbf{x} = 1,$$

with A a real symmetric matrix, show that there is an orthonormal basis such that, if we use coordinates (u, v, w) with respect to this new basis, Σ takes the form

$$\lambda u^2 + \mu v^2 + \nu w^2 = 1.$$

Find λ, μ and ν and hence find the minimum distance between the origin and Σ. *Hint: it is **not** necessary to find the basis explicitly.*

15. (i) Explain what is meant by saying that a 2×2 real matrix,

$$\mathsf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

preserves the scalar product on $\mathbb{R}^2$ with respect to

$$\text{(a) the Euclidean metric, } \mathsf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \text{or (b) the Minkowski metric, } \mathsf{J} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(ii) Using a single real parameter together with a choice of sign (± 1), give and justify a description of all matrices, A, that preserve the scalar product with respect to the Euclidean metric. Show that these matrices form a group.

(iii) Using a single real parameter together with a choice of sign (± 1), give and justify a description of all matrices, A with $a > 0$, that preserve the scalar product with respect to the Minkowski metric. Show that these matrices form a group.

(iv) What is the intersection of the above two groups?

Revision Questions

16. Show that a rotation about the z axis through an angle θ corresponds to the matrix

$$\mathsf{R} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Write down a real eigenvector of R and give the corresponding eigenvalue. In the case of a matrix corresponding to a general rotation, how can one find the axis of rotation?

A rotation through 45° about the x-axis is followed by a similar one about the z-axis. Show that the rotation corresponding to their combined effect has its axis inclined at equal angles

$$\cos^{-1}\frac{1}{\sqrt{(5-2\sqrt{2})}}$$

to the x and z axes.

17. Show that the eigenvalues of a real orthogonal matrix have unit modulus and that if λ is an eigenvalue so is λ^*. Hence argue that the eigenvalues of a 3×3 real orthogonal matrix Q must be a selection from

$$+1\,, \quad -1 \quad \text{and} \quad \mathrm{e}^{i\alpha} \ \&\ \mathrm{e}^{-i\alpha}\,.$$

Verify that $\det\mathsf{Q} = \pm 1$. What is the effect of Q on vectors orthogonal to an eigenvector with eigenvalue ± 1?

$*$18. *This is another way of proving* $\det\mathsf{AB} = \det\mathsf{A}\det\mathsf{B}$. *You may wish to stick to the case* $n = 3$.

If $1 \leqslant r, s \leqslant n$, $r \neq s$ and λ is real, let $E(\lambda, r, s)$ be an $n\times n$ matrix with (i,j) entry $\delta_{ij} + \lambda\delta_{ir}\delta_{js}$. If $1 \leqslant r \leqslant n$ and μ is real, let $F(\mu, r)$ be an $n\times n$ matrix with (i,j) entry $\delta_{ij} + (\mu-1)\delta_{ir}\delta_{jr}$.

(i) Give a simple geometric interpretation of the linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ associated with $\mathsf{E}(\lambda, r, s)$ and $\mathsf{F}(\mu, r)$.

(ii) Give a simple account of the effect of pre-multiplying an $n\times m$ matrix by $\mathsf{E}(\lambda, r, s)$ and by $\mathsf{F}(\mu, r)$. What is the effect of post-multiplying an $m\times n$ matrix?

(iii) If A is an $n\times n$ matrix, find $\det(\mathsf{E}(\lambda, r, s)\mathsf{A})$ and $\det(\mathsf{F}(\mu, r)\mathsf{A})$ in terms of $\det\mathsf{A}$.

(iv) Show that every $n\times n$ matrix is the product of matrices of the form $\mathsf{E}(\lambda, r, s)$ and $\mathsf{F}(\mu, r)$ and a diagonal matrix with entries 0 or 1.

(v) Use (iii) and (iv) to show that, if A and B are $n\times n$ matrices, then $\det\mathsf{A}\det\mathsf{B} = \det\mathsf{AB}$.

$*$19. *The object of this exercise is to show why finding eigenvalues of a large matrix is not just a matter of finding a large fast computer.*

Consider the $n\times n$ complex matrix $\mathsf{A} = \{A_{ij}\}$ given by

$$\begin{aligned} A_{j\,j+1} &= 1 \qquad \text{for } 1 \leqslant j \leqslant n-1 \\ A_{n1} &= \kappa^n \\ A_{ij} &= 0 \qquad \text{otherwise,} \end{aligned}$$

where $\kappa \in \mathbb{C}$ is non-zero. Thus, when $n = 2$ and $n = 3$, we get the matrices

$$\begin{pmatrix} 0 & 1 \\ \kappa^2 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \kappa^3 & 0 & 0 \end{pmatrix}.$$

(i) Find the eigenvalues and associated eigenvectors of A for $n = 2$ and $n = 3$.

(ii) By guessing and then verifying your answers, or otherwise, find the eigenvalues and associated eigenvectors of A for for all $n \geqslant 2$.

(iii) Suppose that your computer works to 15 decimal places and that $n = 100$. You decide to find the eigenvalues of A in the cases $\kappa = 2^{-1}$ and $\kappa = 3^{-1}$. Explain why at least one (and more probably) both attempts will deliver answers which bear no relation to the true answers.