

Mathematical Tripos Part IA

C. P. Caulfield

Differential Equations A3

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DIFFERENTIAL EQUATIONS

Examples Sheet 1

The starred questions are intended as extras: do them if you have time, but not at the expense of unstarred questions on later sheets.

1. Show, from first principles, that, for non-negative integer n , $\frac{d}{dx}x^n = nx^{n-1}$.
2. Let $f(x) = u(x)v(x)$. Use the definition of the derivative of a function to show that

$$\frac{df}{dx} = u \frac{dv}{dx} + \frac{du}{dx}v.$$

3. Calculate
 - (i) $\frac{d^n}{dx^n}(x \cos x)$ using (a) the Leibniz rule and (b) application of the product rule,
 - (ii) $\frac{d^5}{dx^5}([\ln(x)]^2)$.
4. (i) Write down or determine the Taylor series for $f(x) = e^{ax}$ about $x = 1$.
(ii) Write down or determine the Taylor series for $\ln(1+x)$ about $x = 0$. Then show that

$$\lim_{k \rightarrow \infty} k \ln(1 + x/k) = x$$

and deduce that

$$\lim_{k \rightarrow \infty} (1 + x/k)^k = e^x.$$

What property of the exponential function did you need?

5. Determine by any method the first three non-zero terms of the Taylor expansions about $x = 0$ of

(i) $(x^2 + a)^{-3/2}$,

(ii) $\ln(\cos x)$,

iii) $\exp\left\{-\frac{1}{(x-a)^2}\right\}$,

where a is a constant.

6. By considering the area under the curves $y = \ln x$ and $y = \ln(x - 1)$, show that

$$N \ln N - N < \ln(N!) < (N + 1) \ln(N + 1) - N.$$

Hence show that

$$\frac{N!}{N + 1} < \left(\frac{N + 1}{e}\right)^N.$$

7. Show that $y(x) = \int_x^\infty e^{-t^2} dt$ satisfies the differential equation $y'' + 2xy' = 0$.

*8. Let J_n be the indefinite integral

$$J_n = \int \frac{x^{-n} dx}{(ax^2 + 2bx + c)^{\frac{1}{2}}}.$$

By integrating $\int x^{-n-1}(ax^2 + 2bx + c)^{\frac{1}{2}} dx$ by parts, show that for $n \neq 0$,

$$ncJ_{n+1} + (2n - 1)bJ_n + (n - 1)aJ_{n-1} = -x^{-n}(ax^2 + 2bx + c)^{\frac{1}{2}}.$$

Hence evaluate

$$\int_1^2 \frac{dx}{x^{5/2}(x + 2)^{\frac{1}{2}}}.$$

- *9. In a large population, the proportion with income between x and $x + dx$ is $f(x)dx$. Express the mean (average) income μ as an integral, assuming that any positive income is possible.

Let $p = F(x)$ be the proportion of the population with income less than x , and $G(x)$ be the mean (average) income earned by people with income less than x . Further, let $\theta(p)$ be the proportion of the total income which is earned by people with income less than x as a function of the proportion p of the population which has income less than x . Express $F(x)$ and $G(x)$ as integrals and thence derive an expression for $\theta(p)$, showing that

$$\theta(0) = 0, \quad \theta(1) = 1$$

and

$$\theta'(p) = \frac{F^{-1}(p)}{\mu}, \quad \theta''(p) = \frac{1}{\mu f(F^{-1}(p))} > 0.$$

Sketch the graph of a function $\theta(p)$ with these properties. Deduce that, if there is any variation in income, the bottom, (when ordered in terms of income) proportion p of the population receive less than p of the total income, for all positive values of p . Just how much less is quantified by the (in)famous ‘‘Gini index’’ beloved of economists, which is twice the area between the curve $\theta(p)$ and the diagonal line connecting $(0, 0)$ and $(1, 1)$.

10. For $f(x, y) = \exp(-xy)$, find $(\partial f / \partial x)_y$ and $(\partial f / \partial y)_x$. Check that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$. Find $(\partial f / \partial r)_\theta$ and $(\partial f / \partial \theta)_r$,

(i) using the chain rule,

(ii) by first expressing f in terms of the polar coordinates r, θ ,

and check that the two methods give the same results.

[Recall: $x = r \cos \theta$, $y = r \sin \theta$.]

11. If $xyz + x^3 + y^4 + z^5 = 0$ (an implicit equation for any of the variables x, y, z in terms of the other two), find

$$\left(\frac{\partial x}{\partial y}\right)_z, \quad \left(\frac{\partial y}{\partial z}\right)_x, \quad \left(\frac{\partial z}{\partial x}\right)_y$$

and show that their product is -1 .

Does this result hold for an arbitrary relation $f(x, y, z) = 0$?

What about $f(x_1, x_2, \dots, x_n) = 0$?

12. In thermodynamics, the pressure of a system, p , can be considered as a function of the variables V (volume) and T (temperature) or as a function of the variables V and S (entropy).

(i) By expressing $p(V, S)$ in the form $p(V, S(V, T))$ evaluate

$$\left(\frac{\partial p}{\partial V}\right)_T - \left(\frac{\partial p}{\partial V}\right)_S \text{ in terms of } \left(\frac{\partial S}{\partial V}\right)_T \text{ and } \left(\frac{\partial S}{\partial p}\right)_V .$$

(ii) Hence, using $TdS = dU + pdV$ (conservation of energy with U the internal energy), show that

$$\left(\frac{\partial \ln p}{\partial \ln V}\right)_T - \left(\frac{\partial \ln p}{\partial \ln V}\right)_S = \left(\frac{\partial(pV)}{\partial T}\right)_V \left[\frac{p^{-1}(\partial U/\partial V)_T + 1}{(\partial U/\partial T)_V}\right] .$$

$$\left[\text{Hint: } \left(\frac{\partial \ln p}{\partial \ln V}\right)_T = \frac{V}{p} \left(\frac{\partial p}{\partial V}\right)_T\right]$$

13. By differentiating I with respect to λ , show that

$$I(\lambda, \alpha) = \int_0^\infty \frac{\sin \lambda x}{x} e^{-\alpha x} dx = \tan^{-1} \frac{\lambda}{\alpha} + c(\alpha).$$

Show that $c(\alpha)$ is constant (independent of α) and hence, by considering the limits $\alpha \rightarrow \infty$ and $\alpha \rightarrow 0$, show that, if $\lambda > 0$,

$$\int_0^\infty \frac{\sin \lambda x}{x} dx = \frac{\pi}{2}.$$

What is the value of the integral when $\lambda < 0$?

14. Let $f(x) = \left[\int_0^x e^{-t^2} dt\right]^2$ and let $g(x) = \int_0^1 [e^{-x^2(t^2+1)}/(1+t^2)] dt$.

Show that

$$f'(x) + g'(x) = 0.$$

Deduce that

$$f(x) + g(x) = \pi/4,$$

and hence that

$$\int_0^\infty e^{-t^2} dt = \frac{\sqrt{\pi}}{2}.$$

Comments and corrections may be sent by email to cpc12@cam.ac.uk