Mathematical Tripos Part IA **Michaelmas Term 2023**

Differential Equations **Prof. Anthony Challinor**

Examples Sheet 2

The starred questions are intended as extras: do them if you have time, but not at the expense of unstarred questions on later sheets.

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1. According to Newton's law of cooling, the rate of change of the temperature of an object is proportional to the difference in temperature between the object and its surroundings. A forensic scientist enters a crime scene at 5:00 pm and discovers a cup of tea at temperature 40° C. At 5:30 pm its temperature is only 30° C. Giving all details of the mathematical methodology employed and assumptions made, estimate the time at which the tea was made.

2. Determine the half-life of thorium-234 if a sample of mass 5 g is reduced to 4 g in one week. What amount of thorium is left after three months?

3. Find the solutions of the initial-value problems:

(i) $y' + 2y = e^{-x}\,, \qquad y(0) = 1\,;$

(ii) $y' - y = 2xe^{2x}\,, \qquad y(0) = 1\,.$

4. Show that the general solution of

$$y' - y = e^{ux}\,, \tag{$*$}$$

for $u \neq 1$, can be written (by means of a suitable choice of A) in the form

$$y(x) = Ae^{x} + \frac{e^{ux} - e^{x}}{u - 1}\,.$$

By taking the limit as $u \to 1$ and using L'Hôpital's rule, find the general solution of $(*)$ when $u = 1$.

5. Solve:

(i) $x \sin x\, y' + (\sin x + x \cos x)y = xe^{x}\,;$

(ii) $\tan x\, y' + y = 1\,;$

(iii) $y' = (e^{y} - x)^{-1}\,.$

6. Find the general solutions of:

(i) $y' = x^2(1+y^2)$;

(ii) $y' = \cos^2 x \cos^2 2y$;

(iii) $y' = (x-y)^2$;

(iv) $(\mathrm{e}^y + x)y' + (\mathrm{e}^x + y) = 0$.

7. Find all solutions of the equation

$$y\frac{dy}{dx} - x = 0\,,$$

and give a sketch showing the solutions. By means of the substitution $y = \ln u - x$, deduce the general solution of

$$(\ln u - x)\frac{du}{dx} - u\ln u = 0\,.$$

Sketch the solutions, starting from your previous sketch and drawing first the lines to which $y = \pm x$ are mapped.

8. In each of the following cases, sketch a few solution curves:

(i) $y' + xy = 1$;

(ii) $y' = x^2 + y^2$;

(iii) $y' = (1-y)(2-y)$.

It might help you to consider values of y' on the axes, or contours of constant y', or the asymptotic behaviour when y is large.

9. (i) Sketch the solution curves for the equation

$$\frac{dy}{dx} = xy\,.$$

Find the family of solutions determined by this equation and reassure yourself that your sketches were appropriate.

(ii) Sketch the solution curves for the equation

$$\frac{dy}{dx} = \frac{x-y}{x+y}\,. \qquad (*)$$

By rewriting the equation in the form

$$\left(x\frac{dy}{dx} + y\right) + y\frac{dy}{dx} = x\,,$$

find and sketch the family of solutions.

$*$Does the substitution $y = ux$ lead to an easier method of solving $(*)$?

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10. Measurements on a yeast culture have shown that the rate of increase of the amount, or 'biomass', of yeast is related to the biomass itself by the equation

$$\frac{dN}{dt} = aN - bN^2 \,,$$

where $N(t)$ is a measure of the biomass at time t, and a and b are positive constants. Without solving the equation, find in terms of a and b:

(i) the value of N at which dN/dt is a maximum; and

(ii) the values of N at which dN/dt is zero.

Using this information, sketch graphs of $N(t)$ against t, and compare these with what you obtain by solving the equation analytically for $0 \leq N \leq a/b$.

11. Water flows into a deep cylindrical bucket of cross-sectional area A at a volume flow rate Q, which is constant. There is a hole in the bottom of the bucket of cross-sectional area $a \ll A$. When the water level above the hole is h, the volume flow rate out of the hole is $a\sqrt{2gh}$, where g is the gravitational acceleration. Derive an equation for dh/dt. Find the equilibrium depth h_{e} of water, and show that it is stable.

12. In each of the following equations for $y(t)$, find the equilibrium points and classify their stability properties:

(i) $dy/dt = y(y-1)(y-2)\,;$

(ii) $dy/dt = -2\tan^{-1}[y/(1+y^2)]\,;$

$*$(iii) $dy/dt = y^3(\mathrm{e}^y - 1)^2\,.$

13. Investigate the stability of the constant solutions ($u_{n+1} = u_n$) of the discrete equation

$$u_{n+1} = 4u_n(1-u_n)\,.$$

In the case $0 \leq u_0 \leq 1$, use the substitution $u_0 = \sin^2\theta$ to find the general solution and verify your stability results. Can you find an explicit form of the general solution in the case $u_0 > 1$?

$*$**14**. Two identical snowploughs plough the same stretch of road in the same direction. The first starts at $t = 0$ when the (uniform) depth of snow is h_0 and the second starts from the same point at time T later. Snow falls so that the depth of snow increases at a constant rate k. The speed of each snowplough is $k/(ah)$, where h is the depth of snow it is ploughing and a is a constant, and each snowplough clears all the snow. Show that the time taken for the first snowplough to travel a distance x from its starting point, $t_1(x)$, is

$$t_1(x) = \frac{h_0}{k}\left(e^{ax} - 1\right)\,.$$

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Show also that the time $t_2(x)$ by which the second snowplough has travelled a distance x satisfies the equation

$$\frac{1}{a}\frac{dt_2}{dx} = t_2 - \frac{h_0}{k}\left(e^{ax} - 1\right) .$$

Hence show that the snowploughs will collide when they have moved a distance $kT/(ah_0)$.

Comments and corrections may be sent by email to a.d.challinor@ast.cam.ac.uk

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