Mathematical Tripos Part IA | **Lent Term 2017**
Vector Calculus | **Prof R. Jozsa**

Example Sheet 2

1 A circular helix is given by

$$\mathbf{r}(u) \;=\; (\, a\cos u,\, a\sin u,\, cu\,)\,.$$

Calculate the tangent $\mathbf{t}$, curvature κ, principal normal $\mathbf{n}$, binormal $\mathbf{b}$, and torsion τ.

2 Show that a curve in the plane, $\mathbf{r}(t) = (\, x(t),\, y(t),\, 0\,)$, has curvature

$$\kappa(t) \;=\; |\,\dot{x}\ddot{y} - \dot{y}\ddot{x}\,| \,/\, (\,\dot{x}^2 + \dot{y}^2\,)^{3/2}\;.$$

Find the minimum and maximum curvature of the ellipse $x^2/a^2 + y^2/b^2 = 1$ $(a > b > 0)$.

3 Let $\psi(\mathbf{x})$ be a scalar field and $\mathbf{v}(\mathbf{x})$ a vector field. Show, using index notation, that

$$\boldsymbol{\nabla}\cdot(\psi\mathbf{v}) \;=\; (\boldsymbol{\nabla}\psi)\cdot\mathbf{v} \,+\, \psi\,\boldsymbol{\nabla}\cdot\mathbf{v}\,, \qquad \boldsymbol{\nabla}\times(\psi\mathbf{v}) \;=\; (\boldsymbol{\nabla}\psi)\times\mathbf{v} \,+\, \psi\,\boldsymbol{\nabla}\times\mathbf{v}\,.$$

Evaluate (using index notation where necessary) the divergence and the curl of the following:

$$r\,\mathbf{x}\,, \quad \mathbf{a}(\mathbf{x}\cdot\mathbf{b})\,, \quad \mathbf{a}\times\mathbf{x}\,, \quad \mathbf{x}/r^3\,,$$

where $r = |\mathbf{x}|$, and $\mathbf{a}$ and $\mathbf{b}$ are fixed vectors.

4 Use suffix notation to show that

$$\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{v}) \;=\; \mathbf{u}(\boldsymbol{\nabla}\cdot\mathbf{v}) \,+\, (\mathbf{v}\cdot\boldsymbol{\nabla})\mathbf{u} \,-\, \mathbf{v}(\boldsymbol{\nabla}\cdot\mathbf{u}) \,-\, (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{v}\,.$$

for vector fields $\mathbf{u}$ and $\mathbf{v}$. Show also that $(\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u} \;=\; \boldsymbol{\nabla}(\tfrac{1}{2}u^2) - \mathbf{u}\times(\boldsymbol{\nabla}\times\mathbf{u})$.

5 Check, by calculating its curl, that the force field

$$\mathbf{F} \;=\; \left(\, 3x^2\tan z - y^2 \mathrm{e}^{-xy^2}\sin y\;,\; (\cos y - 2xy\sin y)\,\mathrm{e}^{-xy^2}\;,\; x^3\sec^2 z\,\right)$$

is *conservative*. Find the most general scalar potential for $\mathbf{F}$ and hence, or otherwise, find the work done by the force as it acts on a particle moving from $(0,0,0)$ to $(1,\pi/2,\pi/4)$.

6 Verify that the vector field

$$\mathbf{u} \;=\; e^x(x\cos y + \cos y - y\sin y)\,\mathbf{i} \,+\, e^x(-x\sin y - \sin y - y\cos y)\,\mathbf{j}$$

is *irrotational* and express it as the gradient of a scalar field ϕ. Check that $\mathbf{u}$ is also *solenoidal* and show that it can be written as the curl of a vector field $\psi\mathbf{k}$, for some function ψ.

7 **(a)** The vector field $\mathbf{B}(\mathbf{x})$ is everywhere parallel to the normals to a family of surfaces $f(\mathbf{x}) = \text{constant}$. Show that

$$\mathbf{B}\cdot(\boldsymbol{\nabla}\times\mathbf{B}) \;=\; 0\,.$$

7 **(b)** The tangent vector at each point on a curve is parallel to a non-vanishing vector field $\mathbf{H}(\mathbf{x})$. Show that the curvature of the curve is given by $|\mathbf{H}|^{-3}\,|\,\mathbf{H}\times(\mathbf{H}\cdot\boldsymbol{\nabla})\mathbf{H}\,|$.

8 Consider the line integral

$$\oint_C -x^2y\,\mathrm{d}x \,+\, xy^2\,\mathrm{d}y$$

for C a closed curve traversed anti-clockwise in the xy plane.
(i) Evaluate this integral when C is a circle with radius R and centre the origin. Use Green's Theorem to relate the results for $R = b$ and $R = a$ to an area integral over the region $a^2 \leqslant x^2+y^2 \leqslant b^2$, and calculate the area integral directly.
(ii) Now suppose that C is the boundary of a square, with centre the origin, and sides of length ℓ. Show that the line integral is independent of the orientation of the square in the plane.

9 Verify Stokes's Theorem for the hemispherical surface $r = 1$, $z \geqslant 0$, and the vector field

$$\mathbf{F}(\mathbf{r}) \,=\, (\,y,\, -x,\, z\,)\,.$$

10 Let $\mathbf{F}(\mathbf{r}) = (\,x^3{+}3y{+}z^2,\; y^3,\; x^2{+}y^2{+}3z^2\,)$, and let S be the *open* surface

$$1 - z \,=\, x^2 + y^2, \qquad 0 \leqslant z \leqslant 1\,.$$

Use the divergence theorem (and cylindrical polar coordinates) to evaluate $\int_S \mathbf{F}\cdot\mathrm{d}\mathbf{S}$.

Verify your result by calculating the integral directly. [You should find that the vector area element is $\mathrm{d}\mathbf{S} = (\,2\rho\cos\phi\,,\, 2\rho\sin\phi\,,\, 1\,)\,\rho\,\mathrm{d}\rho\,\mathrm{d}\phi\,$.]

11 By applying the divergence theorem to the vector field $\mathbf{a}\times\mathbf{A}$, where $\mathbf{a}$ is an arbitrary constant vector and $\mathbf{A}(\mathbf{x})$ is a vector field, show that

$$\int_V \boldsymbol{\nabla}\times\mathbf{A}\,\mathrm{d}V = -\int_S \mathbf{A}\times\mathrm{d}\mathbf{S}\,,$$

where the surface S encloses the volume V.

Verify this result when S is the sphere $|\mathbf{x}| = R$ and $\mathbf{A} = (z, 0, 0)$ in Cartesian coordinates.

12 By applying Stokes's theorem to the vector field $\mathbf{a}\times\mathbf{F}$, where $\mathbf{a}$ is an arbitrary constant vector and $\mathbf{F}(\mathbf{x})$ is a vector field, show that

$$\oint_C \mathrm{d}\mathbf{x}\times\mathbf{F} \,=\, \int_S (\mathrm{d}\mathbf{S}\times\boldsymbol{\nabla})\times\mathbf{F}\,,$$

where the curve C bounds the open surface S.

Verify this result when C is the unit square in the xy plane with opposite vertices at $(0,0,0)$ and $(1,1,0)$ and $\mathbf{F}(\mathbf{x}) = \mathbf{x}$.

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