Mathematical Tripos Part IA **Lent Term 2017**
Vector Calculus **Prof R Jozsa**

Example Sheet 4

1 The current J_i due to an electric field E_i is given by $J_i = \sigma_{ij} E_j$, where σ_{ij} is the conductivity tensor. In a certain coordinate system,

$$(\sigma_{ij}) = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}.$$

Show that there is a direction along which no current flows, and find the direction(s) along which the current flow is largest, for an electric field of fixed magnitude.

2 Given vectors $\mathbf{u} = (1, 0, 1)$, $\mathbf{v} = (0, 1, -1)$ and $\mathbf{w} = (1, 1, 0)$, find all components of the second-rank and third-rank tensors defined by

$$T_{ij} = u_i v_j + v_i w_j \ ; \qquad S_{ijk} = u_i v_j w_k - v_i u_j w_k + v_i w_j u_k - w_i v_j u_k + w_i u_j v_k - u_i w_j v_k \ .$$

3 Using the transformation law for a second-rank tensor T_{ij}, show that the quantities

$$\alpha = T_{ii}\,, \quad \beta = T_{ij}T_{ji}\,, \quad \gamma = T_{ij}T_{jk}T_{ki}$$

are the same in all Cartesian coordinate systems. If T_{ij} is diagonal in some coordinate system, express the quantities above in terms of its eigenvalues. Hence deduce that the eigenvalues are roots of the cubic equation

$$\lambda^3 - \alpha\lambda^2 + \tfrac{1}{2}(\alpha^2 - \beta)\lambda - \tfrac{1}{6}(\alpha^3 - 3\alpha\beta + 2\gamma) = 0 \ .$$

4 If $u_i(\mathbf{x})$ is a vector field, show that $\partial u_i/\partial x_j$ transforms as a second-rank tensor.
If $\sigma_{ij}(\mathbf{x})$ is a second-rank tensor field, show that $\partial \sigma_{ij}/\partial x_j$ transforms as a vector.

5 The fields $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ obey Maxwell's equations with zero charge and current. Show that the Poynting vector $\mathbf{P} = \mu_0^{-1}\, \mathbf{E}\times\mathbf{B}$ satisfies

$$\frac{1}{c^2}\frac{\partial P_i}{\partial t} + \frac{\partial T_{ij}}{\partial x_j} = 0 \quad \text{where} \quad T_{ij} = \tfrac{1}{2}\,\epsilon_0\,\delta_{ij}\,(\,E_kE_k + c^2B_kB_k\,) - \epsilon_0\,(\,E_iE_j + c^2B_iB_j\,)\ .$$

6 The velocity field $\mathbf{u}(\mathbf{x}, t)$ of an inviscid compressible gas obeys

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\mathbf{u}) = 0 \qquad \text{and} \qquad \rho\Big(\frac{\partial\mathbf{u}}{\partial t} + (\mathbf{u}\cdot\boldsymbol{\nabla})\mathbf{u}\Big) = -\boldsymbol{\nabla}p$$

where $\rho(\mathbf{x}, t)$ is the density and $p(\mathbf{x}, t)$ is the pressure. Show that

$$\frac{\partial}{\partial t}(\,\tfrac{1}{2}\rho u^2\,) + \frac{\partial}{\partial x_i}(\,\tfrac{1}{2}\rho u^2 u_i + p u_i\,) = p\boldsymbol{\nabla}\cdot\mathbf{u} \qquad \text{and} \qquad \frac{\partial}{\partial t}(\,\rho u_i\,) + \frac{\partial}{\partial x_j}(\,t_{ij}\,) = 0$$

for a suitable symmetric tensor t_{ij}, to be determined.

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7 The components of a second-rank tensor are given by a matrix A. Show that

$$A\mathbf{x} = \alpha\mathbf{x} + \boldsymbol{\omega}\times\mathbf{x} + B\mathbf{x} \quad \text{for all } \mathbf{x}\,,$$

for some scalar α, vector $\boldsymbol{\omega}$, and symmetric traceless matrix B. Find α, $\boldsymbol{\omega}$ and B when

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 1 & 2 & 3 \end{pmatrix}.$$

8 **(a)** A tensor of rank 3 satisfies $T_{ijk} = T_{jik}$ and $T_{ijk} = -T_{ikj}$. Show that $T_{ijk} = 0$.

(b) A tensor of rank 4 satisfies $T_{jik\ell} = -T_{ijk\ell} = T_{ij\ell k}$ and $T_{ijij} = 0$. Show that

$$T_{ijk\ell} = \varepsilon_{ijp}\,\varepsilon_{k\ell q}\,S_{pq}\,, \qquad \text{where} \qquad S_{pq} = -T_{rqrp}\,,$$

9 A cuboid of uniform density and mass M has sides of lengths $2a$, $2b$ and $2c$. Find the inertia tensor about its centre, with respect to a coordinate system of your choice.

A cube with sides of length $2a$ has uniform density, mass M, and is rotating with angular velocity $\boldsymbol{\omega}$ about an axis which passes through its centre and through a pair of opposite vertices. What is its angular momentum?

10 Evaluate the following integrals over all space, where $\gamma > 0$ and $r^2 = x_p x_p$:

$$\text{(i)} \quad \int r^{-3} e^{-\gamma r^2} x_i x_j \,\mathrm{d}V\,; \qquad \text{(ii)} \quad \int r^{-5} e^{-\gamma r^2} x_i x_j x_k \,\mathrm{d}V\,.$$

11 A tensor has components T_{ij} with respect to Cartesian coordinates x_i. If the tensor is invariant under arbitrary rotations around the x_3-axis, show that it must have the form

$$(T_{ij}) = \begin{pmatrix} \alpha & \omega & 0 \\ -\omega & \alpha & 0 \\ 0 & 0 & \beta \end{pmatrix}.$$

12 In linear elasticity, the symmetric second-rank stress tensor σ_{ij} depends on the symmetric second-rank strain tensor e_{kl} according to $\sigma_{ij} = c_{ijkl}e_{kl}$. Explain why c_{ijkl} must be a fourth-rank tensor, assuming $c_{ijkl} = c_{ijlk}$. For an isotropic medium, use the most general possible form for c_{ijkl} (which you may quote) to show that

$$\sigma_{ij} = \lambda\delta_{ij}e_{kk} + 2\mu e_{ij}\,,$$

where λ and μ are scalars.

Invert this equation to express e_{ij} in terms of σ_{ij}, assuming $\mu \neq 0$ and $3\lambda \neq -2\mu$. Explain why the principal axes of σ_{ij} and e_{ij} coincide.

The elastic energy density resulting from a deformation of the medium is $E = \frac{1}{2}e_{ij}\sigma_{ij}$. Show that E is strictly positive for any non-zero strain e_{ij} provided $\mu > 0$ and $\lambda > -2\mu/3$.

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