Electromagnetism: Example Sheet 2

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1. A constant magnetic field points along the z-axis: $\mathbf{B} = B\hat{\mathbf{z}}$. Verify that each of the following vector potentials satisfies $\mathbf{B} = \nabla \times \mathbf{A}$:

- $\mathbf{A} = xB\hat{\mathbf{y}}$,
- $\mathbf{A} = \frac{1}{2}(xB\hat{\mathbf{y}} - yB\hat{\mathbf{x}})$,
- In cylindrical polar coordinates, $\mathbf{A} = \frac{1}{2}rB\hat{\boldsymbol{\phi}}$, with $r^2 = x^2 + y^2$,
- In spherical polar coordinates, $\mathbf{A} = \frac{1}{2}r\sin\theta\, B\hat{\boldsymbol{\phi}}$, with $r^2 = x^2 + y^2 + z^2$.

2. A cylindrical conductor of radius a, with axis along the z-axis, carries a uniform current density $\mathbf{J} = J\hat{\mathbf{z}}$. Use Ampère's law to show that the magnetic field within the conductor is given, in cylindrical polar coordinates, by

$$\mathbf{B} = \frac{1}{2}\mu_0 J r\hat{\boldsymbol{\phi}}$$

with $r^2 = x^2 + y^2$. [In this question, and the following question, you may assume that the magnetic field inside a conductor is the same as in a vacuum.]

3. A steady current I flows in the z-direction uniformly in the the region between the cylinders $x^2 + y^2 = a^2$ and $(x + d)^2 + y^2 = b^2$, where $0 < d < (b - a)$. Show that the associated magnetic field $\mathbf{B}$ throughout the region $x^2 + y^2 < a^2$ is given by

$$\mathbf{B} = \frac{\mu_0 I d}{2\pi(b^2 - a^2)}\hat{\mathbf{y}}$$

4. Use the Biot-Savart law to determine the magnetic field:

- Around an infinite, straight wire carrying current I.
- At the centre of a square loop of wire, with sides of length a, carrying current I.
- At the point $(0, 0, z)$ above a loop of wire of radius a, lying in the (x, y) plane, with centre at the origin, carrying current I.

5. Explain why the force $\mathbf{F}$ and torque $\boldsymbol{\tau}$ experienced by a loop of wire C carrying current I are given by

$$\mathbf{F} = I\oint_C d\mathbf{r} \times \mathbf{B} \quad \text{and} \quad \boldsymbol{\tau} = I\oint_C \mathbf{r} \times (d\mathbf{r} \times \mathbf{B})$$

A loop of wire lies in a plane whose normal makes an angle θ with a uniform magnetic field. The loop of wire encloses a planar area A and carries current I. Compute the torque.

6. What boundary conditions apply on either side of a surface current $\mathbf{K}$?

A surface current experiences a Lorentz force from the *average* magnetic field on either side of the surface. A wire carrying current I winds N times per unit length to form a cylindrical solenoid. Show that there is a force per unit area on the cylinder (outward normal $\hat{\mathbf{n}}$) given by

$$\mathbf{f} = \frac{\mu_0 I^2 N^2}{2}\hat{\mathbf{n}}$$

.

7*. A steady current I_1 flows around a closed loop C_1. Use the Biot-Savart law to show that this exerts a force on a second loop C_2 carrying current I_2, given by

$$\mathbf{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_2} \oint_{C_1} d\mathbf{r}_2 \times \left(d\mathbf{r}_1 \times \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \right)$$

Write this in a form which exhibits anti-symmetry, $\mathbf{F}_{12} = -\mathbf{F}_{21}$ (i.e. Newton's third law).

8. A current creates a time-dependent electric and magnetic field given, in cylindrical polar coordinates, by

$$\mathbf{E} = e^{-t}\hat{\boldsymbol{\phi}} \quad , \quad \mathbf{B} = \frac{e^{-t}}{r}\hat{\mathbf{z}}$$

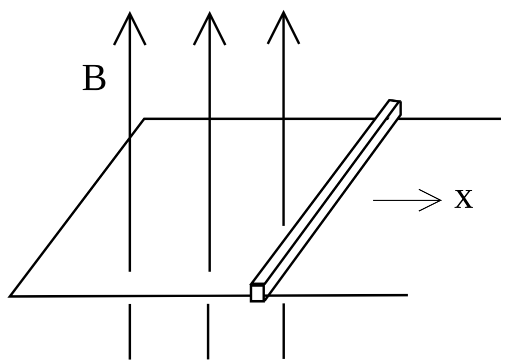
(Here $r^2 = x^2 + y^2$). Verify that these are consistent with the remaining Maxwell equations $\nabla \cdot \mathbf{E} = \nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + \partial \mathbf{B}/\partial t = 0$.

The emf around a moving circuit $C(t)$ is given by

$$\oint_{C(t)} (\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot d\mathbf{r} = -\frac{d}{dt} \int_{S(t)} \mathbf{B} \cdot d\mathbf{S}$$

where $S(t)$ is the surface spanning $C(t)$ and $\mathbf{v}$ is the velocity of a point on the circuit. Verify this equation by explicitly evaluating the integrals for a circle $C(t)$ lying in the plane $z = 0$ with radius $R(t) = 1 + t$ in the electric and magnetic fields given above.

9. A horizontal, rectangular circuit, shown in the figure, has a sliding bar of mass m and length L which moves, without friction, in the x-direction. The bar and all the wires in the circuit have resistance R per unit length.



A uniform vertical magnetic field $\mathbf{B} = (\alpha/t)\hat{\mathbf{z}}$ is applied for time $t > 0$, with α constant. Derive the differential equation satisfied by the position x for $t > 0$. Find a solution.

[In this question, and the following question, you may assume that the effect on the magnetic field due to any current flow is negligible compared to the background $\mathbf{B}$.]

10. A vector potential is given, in cylindrical polar coordinates, by $A_\phi = \frac{1}{2}Brz$ where B is constant (and, again, $r^2 = x^2 + y^2$). Compute the magnetic field $\mathbf{B}$.

A conducting loop of radius a and resistance R lies in the (x, y) plane at position $z(t)$, its centre on the axis. Find the induced current in the loop.

Compute the force exerted on the loop by the magnetic field. To overcome this, an equal and opposite force is applied to the loop. Show that the work done per unit time by this force is equal to the rate of dissipation of energy due to the resistance in the loop.