MATHEMATICAL TRIPOS Part IB **A. J. Macfarlane**

ELECTROMAGNETISM

Lent 2004

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VECTOR CALCULUS REVISION

1. Vector Algebra in $\mathbb{R}^3$

Suffix notation. Use of δ_{ij} and ϵ_{ijk}. (a_i represents $\mathbf{a}$, etc.)

$$\delta_{ij}a_j = a_i, \quad \delta_{kk} = 3$$

$$\epsilon_{ijk}\epsilon_{pqk} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}, \qquad \epsilon_{ijk}\epsilon_{pjk} = 2\delta_{ip}$$

$$\mathbf{a}\cdot\mathbf{b} = a_i\delta_{ij}b_j = a_jb_j, \qquad (\mathbf{a}\wedge\mathbf{b})_k = \epsilon_{kpq}a_pb_q$$

2. Vector Calculus in $\mathbb{R}^3$

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z} = \sum_{i=1}^{3}\mathbf{e}_i\frac{\partial}{\partial r_i},$$

where $x = r_1$, $\mathbf{e}_1 = \mathbf{i}$, etc.

Let $\mathbf{r}$ or r_i denote the position vector, and $r = |\mathbf{r}|$. Then $\hat{\mathbf{r}} = \mathbf{r}/r$ defines a unit vector in the direction of $\mathbf{r}$. We note a key result (1), and some consequences (for $r \neq 0$)

$$\partial_i r_j = \delta_{ij} \tag{1}$$

$$\partial_i r = \frac{r_i}{r}, \quad \text{or} \quad \nabla r = \hat{\mathbf{r}}\,; \quad \frac{\partial}{\partial r_i}\left(\frac{1}{r}\right) = -\frac{r_i}{r^3}, \tag{2a}$$

$$\frac{\partial^2}{\partial r_i\partial r_j}\left(\frac{1}{r}\right) = \frac{3r_ir_j - r^2\delta_{ij}}{r^5}, \quad \nabla^2\left(\frac{1}{r}\right) = 0 \tag{2b}$$

Let $\mathbf{a}(\mathbf{r})$, $\mathbf{b}(\mathbf{r})$ and $\phi(\mathbf{r})$ denote vector, vector and scalar fields. Then

$$(\nabla\phi)_k = \partial_k\phi, \quad \nabla\cdot\mathbf{a} = \partial_j a_j, \quad (\nabla\wedge\mathbf{a})_k = \epsilon_{kpq}a_pb_q \tag{3}$$

Sometimes one writes $\operatorname{grad}\phi = \nabla\phi$, $\operatorname{div}\mathbf{a} = \nabla\cdot\mathbf{a}$, and $\operatorname{curl}\mathbf{a} = \nabla\wedge\mathbf{a}$.

3. Vector Calculus identities in $\mathbb{R}^3$

$$\nabla\cdot(\phi\mathbf{a}) = \mathbf{a}\cdot\nabla\phi + \phi\nabla\cdot\mathbf{a} \tag{4}$$

$$\nabla\wedge(\phi\mathbf{a}) = \mathbf{a}\wedge\nabla\phi - \phi\nabla\wedge\mathbf{a} \tag{5}$$

$$\nabla \cdot \mathbf{a}\wedge\mathbf{b} = \mathbf{b} \cdot \nabla\wedge \mathbf{a} - \mathbf{a} \cdot \nabla\wedge \mathbf{b} \tag{6}$$

There are similar but more complicated identities for $\nabla\wedge(\mathbf{a}\wedge\mathbf{b})$ and for $\nabla(\mathbf{a}\cdot\mathbf{b})$. The simple examples of these which crop up, often when one of the vector fields involved is actually constant, are best handled by direct suffix notation methods.

$$\nabla^2\phi = \partial_k\partial_k\phi \tag{7}$$

$$\nabla\wedge\nabla\phi = 0, \quad \nabla \cdot (\nabla\wedge \mathbf{a}) = (\nabla\wedge\nabla) \cdot \mathbf{a} = 0 \tag{8}$$

$$\nabla\wedge(\nabla\wedge\mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2\mathbf{a} \tag{9}$$

A warning (that may not be needed in the context of the present course). In Cartesian coordinates we have

$$(\nabla^2\mathbf{a})_j = \nabla^2\mathbf{a}_j$$

using $\nabla^2 = \partial_k\partial_k$. But the same does not hold in other coordinate systems because the unit vectors in the explicit expression for $\mathbf{a}$ in terms of the corresponding unit vectors as basis, are coordinate dependent. For all coordinate systems other than Cartesians, one uses (9) in the form

$$\nabla^2\mathbf{a} = \nabla(\nabla \cdot \mathbf{a}) - \nabla\wedge(\wedge\, \mathbf{a}).$$

Each step on the right-hand side here is well-defined in any system of curvilinear coordinates. See Sec. 5 below.

4. Integral Theorems

Divergence Theorem in $\mathbb{R}^3$

$\mathbf{F(r)}$ is a vector field defined in $V \subset \mathbb{R}^3$; V has surface $S = \partial V$, and $d\mathbf{S}$ denotes a surface element parallel to the outward unit normal $\mathbf{n}$. $\quad$ ($\mathbf{n}^2 = 1$.)

$$\int_V \nabla \cdot \mathbf{F} dV = \int_S \mathbf{F} \cdot d\mathbf{S} = \int_S \mathbf{n} \cdot \mathbf{F} dS \tag{10}$$

Stokes's Theorem in $\mathbb{R}^3$

S is an open orientable surface (no Möbius bands), bounded by a closed curve $C = \partial S$, and lies entirely within a simply connected volume within which $\mathbf{F(r)}$ is defined and differentiable. C is traversed in an anticlockwise direction with respect to the unit normal $\mathbf{n}$, to S.

$$\int_S \mathbf{n} \cdot \nabla\wedge \mathbf{F}\, dS = \oint_C \mathbf{F} \cdot d\mathbf{r} \tag{11}$$

Corollaries to these two theorems arise in various ways, *e.g.* by writing $\mathbf{F} = \mathbf{c}\phi$ where $\mathbf{c}$ is an arbitrary constant vector, or $\mathbf{F} = \mathbf{c}\wedge\mathbf{a}(\mathbf{r})$.

5. Expressions for $\nabla\psi$, $\nabla \cdot \mathbf{A}$, $\nabla\wedge \mathbf{A}$, and $\nabla^2\psi$ in curvilinear coordinates

Cylindrical polars

Coordinates (s, ϕ, z) : $x = r\cos\theta$, $y = r\sin\theta$, $z = z$.

$$\nabla\psi = \left(\frac{\partial\psi}{\partial s}, \frac{1}{s}\frac{\partial\psi}{\partial\phi}, \frac{\partial\psi}{\partial z}\right) \tag{12}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_s) + \frac{1}{s}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z} \tag{13}$$

$$\nabla\wedge\mathbf{A} = \left(\frac{1}{s}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}, \frac{\partial A_s}{\partial z} - \frac{\partial A_z}{\partial s}, \frac{1}{s}\frac{\partial}{\partial s}(sA_\phi) - \frac{1}{s}\frac{\partial A_s}{\partial\phi}\right) \tag{14}$$

$$\nabla^2\psi = \frac{1}{s}\frac{\partial}{\partial s}\left(s\frac{\partial\psi}{\partial s}\right) + \frac{1}{s^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2} \tag{15}$$

Let $\mathbf{e}_s$, $\mathbf{e}_\phi$, and $\mathbf{e}_z$ be the unit vectors respectively in the direction of increase of s (at constant ϕ, z), ϕ (at constant z, s), and z (at constant s, ϕ). Then these are a right handed orthonormal triad. Also

$$\nabla\psi = \mathbf{e}_s\frac{\partial\psi}{\partial s} + \mathbf{e}_\phi\frac{1}{s}\frac{\partial\psi}{\partial\phi} + \mathbf{e}_z\frac{\partial\psi}{\partial z}. \tag{16}$$

The unit vectors of cylindrical polars are related to the Cartesian unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, via

$$\mathbf{e}_s = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}, \quad \mathbf{e}_\phi = -\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}, \quad \mathbf{e}_z = \mathbf{k}. \tag{17}$$

Spherical polars

Coordinates (r, θ, ϕ) : $x = r\sin\theta\,\cos\phi$, $y = r\sin\theta\,\sin\phi$, $z = r\cos\theta$.

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\right) = \mathbf{e}_r\frac{\partial\psi}{\partial r} + \mathbf{e}_\theta\frac{1}{r}\frac{\partial\psi}{\partial\theta} + \mathbf{e}_\phi\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi} \tag{18}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi} \tag{19}$$

$$\begin{aligned}\nabla\wedge\mathbf{A} = \Bigg(\frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right), \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{\partial}{\partial r}(rA_\phi)\right), \\ \frac{1}{r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\Bigg)\end{aligned} \tag{20}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} \tag{21}$$

6. Integration over all space V

Cartesians: $\int_V d\tau = \int_{-\infty}^{\infty} dx\,dy\,dz$.

Cylindrical polars : $\int_V d\tau = \int_0^\infty s\,ds\int_0^{2\pi} d\phi\int_0^\infty dz$.

Spherical polars : $\int_V d\tau = \int_0^\infty r^2dr\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\phi$, and

$$\int_V d\tau f(r) = 4\pi\int_0^\infty f(r)r^2dr. \tag{22}$$

Revision examples

These questions cover mainly straightforward material from the IA Vector Calculus course. Some of the questions contain reslts that will be used in the lecture course. The examples are not intended to occupy supervision time.

1. Show that $\mathbf{a}\wedge(\mathbf{b}\wedge\mathbf{c}) + \mathbf{b}\wedge(\mathbf{c}\wedge\mathbf{a}) + \mathbf{c}\wedge(\mathbf{a}\wedge\mathbf{b}) = \mathbf{0}$.

2. Let $\mathbf{a}$ be a constant vector. Use suffix notation methods to check these identities.

$$\nabla(\mathbf{r}\cdot\mathbf{a}) = \mathbf{a}, \quad (\mathbf{a}\cdot\nabla)\mathbf{r} = \mathbf{a}$$

$$\nabla\wedge(\mathbf{a}\wedge\mathbf{F}) = \mathbf{a}\nabla\cdot\mathbf{F} - (\mathbf{a}\cdot\nabla)\mathbf{F}, \quad \text{and} \quad \text{hence} \quad \nabla\wedge(\mathbf{a}\wedge\mathbf{r}) = 2\mathbf{a}$$

$$\nabla\left(\frac{1}{|\mathbf{r}-\mathbf{a}|}\right) = -\frac{(\mathbf{r}-\mathbf{a})}{|\mathbf{r}-\mathbf{a}|^3}$$

$$\partial_i\partial_j\frac{1}{r} = \frac{3r_ir_j - r^2\delta_{ij}}{r^5}$$

3. In the notation of (17), show that

$$\frac{\partial\mathbf{e}_s}{\partial\phi} = \mathbf{e}_\phi, \quad \frac{\partial\mathbf{e}_\phi}{\partial\phi} = -\mathbf{e}_s.$$

4. Show (using an arbitrary constant vector $\mathbf{a}$ for (ii)) that:

(i) $\int_S r^n\mathbf{r}.d\mathbf{S} = \int_V (n+3)r^n dV$, (ii) $\int_S r^n d\mathbf{S} = \int_V nr^{n-2}\mathbf{r}dV$

5. Let S_1 be a closed surface entirely contained within a closed surface S_2. Let V be the volume bounded by S_1 and S_2. If $\nabla\cdot\mathbf{a} = 0$ throughout V, show that $\int_{S_1} d\mathbf{S}\cdot\mathbf{a} = \int_{S_2} d\mathbf{S}\cdot\mathbf{a}$.

6. Let $\mathbf{a}$ be an arbitrary constant vector. Use Stokes's theorem and the fourth result from 2 to show that

$$\tfrac{1}{2}\mathbf{a}\cdot\oint_{C=\partial S}\mathbf{r}\wedge d\mathbf{r} = \mathbf{a}\cdot\int_S d\mathbf{S},$$

so that the area of plane C, with $\mathbf{n}$ the unit normal to the plane, is given by

$$S\mathbf{n} = \tfrac{1}{2}\oint_{C=\partial S}\mathbf{r}\wedge d\mathbf{r}.$$

7. If $\mathbf{B} = (0, 0, B)$ in cartesians with B constant, verify that the following possible vector potentials yield $\mathbf{B} = \nabla\wedge\mathbf{A}$:

(i) in cartesians, $\mathbf{A} = (0, xB, 0)$

(ii) in cylindrical polars, $\mathbf{A} = (0, \tfrac{1}{2}B\,s, 0)$

(iii) in spherical polars, $\mathbf{A} = (0, 0, \tfrac{1}{2}B\,r\sin\theta)$

[For (iii), from a decent diagram of spherical polars, find $\mathbf{k} = \mathbf{e}_r\cos\theta - \mathbf{e}_\theta\sin\theta$.]

8. For $\phi = -Ez = -Er\cos\theta$ calculate $\mathbf{E} = -\nabla\phi$ in Cartesian and spherical polar coordinates, to which applies the hint to 7.