B38
Mathematical Tripos Part IB Prof A.C. Davis
INTRODUCTION TO SPECIAL RELATIVITY (P6) Lent Term 2009

Problems: 1–4 are 2-dimensional, 5–10 are 4-dimensional.

1. In a 2-dimensional spacetime an inertial frame $\mathcal{S}'$ moves with velocity u relative to an inertial frame $\mathcal{S}$. Write down an appropriate Lorentz transformation between $\mathcal{S}$ and $\mathcal{S}'$. A particle p moves with speed v with respect to $\mathcal{S}$ and v' with respect to $\mathcal{S}'$, so that if its position is measured at two successive instants $dx = vdt$ and $dx' = v'dt'$. Suppose the two clocks agree for p, i.e., $dt' = dt$. Show that p is moving with constant velocity and

$$v = \frac{c^2}{u}\left[1 - \sqrt{1 - u^2/c^2}\right].$$

2. Two clocks are at rest in inertial frames $\mathcal{S}$ and $\mathcal{S}'$ whose relative velocity is u in a 2-dimensional spacetime, and the clocks indicate $t = t' = 0$ when the two spatial origins coincide. When the clock in $\mathcal{S}'$ reads $\Delta t'$ it receives a radio signal from the clock in $\mathcal{S}$ sent out at time Δt. Draw a spacetime diagram describing this process. Hence, or otherwise, deduce the Doppler shift equation,

$$\Delta t = \Delta t' \sqrt{\frac{1 - u/c}{1 + u/c}}.$$

3. (a) Rewrite the wave equation for a scalar field,

$$\frac{1}{c^2}\frac{\partial^2 \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2},$$

in terms of the coordinates

$$x_\pm = x \pm ct.$$

(b) Show that under a Lorentz transformation the coordinates transform as follows,

$$x_\pm \to x'_\pm = \sqrt{\frac{c \mp v}{c \pm v}} x_\pm$$

where v is the velocity associated with the transformation. Hence show that if $\phi(x_+, x_-)$ is a solution of the wave equation then $\phi(x'_+, x'_-)$ is also.

*(c) Using the fact that the general solution of the wave equation is

$$\phi = f(x_+) + g(x_-),$$

where f and g are arbitrary twice differentiable functions, derive the Doppler shift formula of question 2.

4. Use a spacetime diagram to demonstrate the phenomenon of length contraction.

In Porterhouse, college maintenance staff are expected to work fast. A man runs at a speed u corresponding to $\gamma = 2$, carrying a $20m$ long ladder, into a shed of length $10m$, where an assistant is then able to close the door! However from the runner's point of view, he encounters a high speed shed of length $5m$, which is able to enclose his $20m$ long ladder. Explain!

This is a typical special relativity "paradox". As stated it involves sharp deaccelerations when the runner meets with the end wall. Therefore replace the shed with a "Dutch barn", (i.e., fixed dimensions but open walls). Compute the velocity parameter $\beta = u/c = \sqrt{3}/2 \approx 0.9$, and deduce $1/\beta \approx 1.1$. Let the barn's inertial frame be $\mathcal{S}$, and the runner's be $\mathcal{S}'$. Assume that the front of the ladder f and the first barn wall s coincide at $t = t' = 0$. Draw spacetime diagrams for both $\mathcal{S}$ and $\mathcal{S}'$, showing clearly the spacetime paths of f and s, as well as the paths of b, the back of the ladder, and e, the other barn wall, and estimate the intercepts of these straight lines with the coordinate axes. Now explain the paradox.

5. In an inertial frame $\mathcal{S}$ a photon with energy E moves in the xy-plane at an angle θ relative to the x-axis. Show that in a second frame $\mathcal{S}'$ whose relative speed is u directed in the x-direction, the energy and angle are given by

$$E' = \gamma E(1 - \beta\cos\theta), \qquad \cos\theta' = \frac{\cos\theta - \beta}{1 - \beta\cos\theta},$$

where $\beta = u/c$ and $\gamma = (1-\beta^2)^{-1/2}$. Write down E and $\cos\theta$ as functions of E' and $\cos\theta'$.

Show that for a photon moving in the x-direction there is a frequency change by a factor $\sqrt{(1-\beta)/(1+\beta)}$ — this is the *special relativistic Doppler effect*.

Next consider a source of photons which is at rest in $\mathcal{S}'$. Consider the photons emitted in the forward direction, i.e., $\cos\theta' > 0$. Show that if β is close to unity, these photons will appear in $\mathcal{S}$ to be concentrated in a narrow cone about $\theta = 0$ — this is the *headlight effect*.

6. Pulsars are stars which emit pulses of radiation at a regular frequency. Jack and Jill are twins who count pulses from a very distant pulsar (thousands of light years away) in the y-direction. She travels at a speed given by $\beta = 24/25$ in the x-direction for seven years and then comes back at the same speed, while he stays at home. At the end of the trip they have counted the same number of pulses. Use question 5 to confirm that on return she has aged by 14 years and he by 50.

7. A body of rest mass m_0 disintegrates at rest into two parts of rest masses m_1 and m_2. Show that the energies of the parts are

$$E_1 = c^2\,\frac{m_0^2 + m_1^2 - m_2^2}{2m_0}, \qquad E_2 = c^2\,\frac{m_0^2 + m_2^2 - m_1^2}{2m_0}.$$

8. Two particles with rest masses m_1 and m_2 scatter elastically. Show that in the centre-of-momentum frame, p_1 and p_2 the 3-momenta before collision and q_1 and q_2, the 3-momenta after collision all lie on a circle, i.e., they are coplanar and all have the same magnitude.

9. A photon (of zero rest mass) collides with an electron of rest mass m_0 which is initially at rest. Show that the angle θ by which the photon is deflected is related to the magnitudes p and q of its initial and final momenta by

$$2\sin^2\tfrac{1}{2}\theta = \frac{m_0c}{q} - \frac{m_0c}{p}.$$

10. In a laboratory frame a particle of rest mass m_1 has energy E_1, and a second particle of rest mass m_2 is at rest. Show that in units where $c = 1$, the combined energy in the centre-of-momentum frame is

$$\sqrt{m_1^2 + m_2^2 + 2E_1m_2}.$$

Hence show that in a collision of one proton with energy E on another one at rest it is possible to create a proton-antiproton pair (in addition to the original protons) if $E \geq 7m$ (where $m_1 = m_2 = m$ is the mass of a proton and of an antiproton).

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