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Example Sheet 1

1. Consider the two-dimensional flow $u = 1/(1+t)$, $v = 1$ in $t > -1$. Find and sketch
(i) the streamline at $t = 0$ which passes through the point $(1, 1)$,
(ii) the path of a fluid particle which is released from $(1, 1)$ at $t = 0$.

2. A steady two-dimensional flow (pure straining) is given by $u = \alpha x$, $v = -\alpha y$ with $\alpha > 0$ constant.
(i) Find the equation for a general streamline of the flow, and sketch some of them.
(ii) At $t = 0$ the fluid on the curve $x^2 + y^2 = a^2$ is marked (by an electro-chemical technique). Find the equation for this material fluid curve for $t > 0$.
(iii) Does the area within the curve change in time, and why?

3. Repeat question 2(ii) for the two-dimensional flow (simple shear) given by $u = \gamma y$, $v = 0$ with $\gamma > 0$ constant. Sketch the streamlines and the material curve at $\gamma t \approx 0, 1, 2$.

4. An incompressible two-dimensional flow is represented by a streamfunction $\psi(x, y)$ with $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Show that
(i) the streamlines are given by $\psi = \text{const}$,
(ii) $|\mathbf{u}| = |\nabla\psi|$, so that the flow is faster where the streamlines are closer,
(iii) the volume flux crossing any curve from $\mathbf{x}_0$ to $\mathbf{x}_1$ is given by $\psi(\mathbf{x}_1) - \psi(\mathbf{x}_0)$,
(iv) $\psi = \text{const}$ on any *fixed* (i.e. stationary) boundary.
[*Hint for (iii):* $\mathbf{n}\, ds = (dy, -dx)$.]

5. Verify that the two-dimensional flow given in Cartesian coordinates by

$$u = \frac{y-b}{(x-a)^2 + (y-b)^2}, \qquad v = \frac{a-x}{(x-a)^2 + (y-b)^2}$$

satisfies $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, and then find the streamfunction $\psi(x, y)$ such that $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$. Sketch the streamlines.

6. Verify that the two-dimensional flow given in polar coordinates by

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta, \qquad u_\theta = -U\left(1 + \frac{a^2}{r^2}\right)\sin\theta$$

satisfies $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$ and find the streamfunction $\psi(r, \theta)$. Sketch the streamlines, starting with $\psi = 0$.

7*. Verify that the axisymmetric flow (uniaxial straining) given in cylindrical polar coordinates by $u_r = -\frac{1}{2}\alpha r$, $u_z = \alpha z$ satisfies $\boldsymbol{\nabla} \cdot \mathbf{u} = 0$, and find the Stokes streamfunction $\Psi(r, z)$. Sketch the streamlines in the (r, z)-plane.

[*For axisymmetric flow in coordinates* (r, θ, z)

$$\boldsymbol{\nabla} \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{\partial u_z}{\partial z}, \quad \textit{and} \quad u_r = -\frac{1}{r}\frac{\partial \Psi}{\partial z}, \quad u_z = \frac{1}{r}\frac{\partial \Psi}{\partial r} \quad \left(\textit{from } A_\theta = \frac{\Psi}{r}\right).]$$

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Comment and advice on the wording of questions in fluid mechanics: In the context of the course, you are expected to assume that water behaves in flows like the ones below as an incompressible, inviscid fluid, though the question often doesn't say so explicitly! Densities, viscosities and other parameters are constants unless stated otherwise. Words like 'small' and 'narrow' often imply use of a suitable approximation.

Note to students and supervisors: Every answer should include a relevant sketch.

8. An axisymmetric jet of water of speed $U = 1\,\mathrm{m\,s^{-1}}$ and cross-section $A = 6 \times 10^{-4}\,\mathrm{m}^2$ strikes a wall at right angles and spreads out over it. By using the momentum integral equation over a suitable control volume, and neglecting gravity, calculate the force on the wall due to the jet.

9. Starting from the Euler momentum equation for an incompressible fluid of density ρ with a potential force $-\boldsymbol{\nabla}\chi$, show that for a fixed volume V enclosed by surface ∂V

$$\frac{d}{dt}\int_V \tfrac{1}{2}\rho u^2\,dV + \int_{\partial V} H\mathbf{u}\cdot\mathbf{n}\,dA = 0\,,$$

where $H = \frac{1}{2}\rho u^2 + p + \chi$ is the Bernoulli quantity, so concluding that $H\mathbf{u}$ is the energy flux and H is the transportable energy. Comment on the interpretation of $\mathbf{u}\cdot\boldsymbol{\nabla}H = 0$ in steady flow.

10. A cylindrical tank of radius a is filled to a depth h_0 with fluid of density ρ. The tank is rotated about its axis with angular velocity Ω for a long time, until the fluid rotates uniformly with it and $\mathbf{u} = (\Omega y, -\Omega x, 0)$. Use the Euler equation and the free-surface boundary condition to determine the pressure distribution $p(r, z)$ and the height of the free surface $h(r)$ for the case $h_0 \geqslant \Omega^2 a^2/4g$. Comment on the physical significance of the term $\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u}$.

11. How high can water rise up one's arm hanging in the river from a lazily moving ($1\,\mathrm{m\,s^{-1}}$) punt? [*Hint: Use Bernoulli on a surface streamline.*]

12. Waste water flows into a large open-topped tank with volume flux Q and out through a small exit pipe of cross-sectional area A into the air. In steady state, how high above the pipe is the water in the tank?

13. A water clock is an axisymmetric vessel with a small exit pipe in the bottom. Find the shape for which the water level falls equal heights in equal intervals of time.

14*. A flat-bottomed barge closely fits a canal, so that while it travels slowly it still generates a fast current with speed U under it. Estimate how much lower the barge sits in the water as a result of this current when $U = 5\,\mathrm{m\,s^{-1}}$. [*Hint: Archimedes when stationary. Flow reduces pressure, so have to sit deeper for same pressure on long bottom.*]

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