A. Shadrin

Mathematical Tripos Part IB: Lent Term 2024

Numerical Analysis – Examples' Sheet 3

1. Calculate *all* LU factorizations of the matrix

$$A = \begin{bmatrix} 10 & 6 & -2 & 1 \\ 10 & 10 & -5 & 0 \\ -2 & 2 & -2 & 1 \\ 1 & 3 & -2 & 3 \end{bmatrix},$$

where all diagonal elements of L are one. By using one of these factorizations, find *all* solutions of the equation $A\boldsymbol{x} = \boldsymbol{b}$ where $\boldsymbol{b}^T = [-2, 0, 2, 1]$.

2. By using column pivoting if necessary to exchange rows of A, an LU factorization of a real $n \times n$ matrix A is calculated, where L has ones on its diagonal, and where the moduli of the off-diagonal elements of L do not exceed one. Let α be the largest of the moduli of the elements of A. Prove by induction on i that elements of U satisfy the condition $|u_{ij}| \le 2^{i-1}\alpha$. Then construct 2×2 and 3×3 nonzero matrices A that yield $|u_{22}| = 2\alpha$ and $|u_{33}| = 4\alpha$ respectively.

3. Let A be a real $n \times n$ matrix that has the factorization $A = LU$, where L is lower triangular with ones on its diagonal and U is upper triangular. Prove that, for every integer $k \in \{1, 2, \ldots, n\}$, the first k rows of U span the same space as the first k rows of A. Prove also that the first k columns of A are in the k-dimensional subspace that is spanned by the first k columns of L. Hence deduce that no LU factorization of the given form exists if we have $\operatorname{rank} H_k < \operatorname{rank} B_k$, where H_k is the leading $k \times k$ submatrix of A and where B_k is the $n \times k$ matrix whose columns are the first k columns of A.

4. Calculate the Cholesky factorization of the matrix

$$\begin{bmatrix} 1 & 1 & & & & \\ 1 & 2 & 1 & & & \\ & 1 & 3 & 1 & & \\ & & 1 & 4 & 1 & \\ & & & 1 & 5 & 1 \\ & & & & 1 & \lambda \end{bmatrix}.$$

Deduce from the factorization the value of λ that makes the matrix singular. Also find this value of λ by seeking the vector in the null-space of the matrix whose first component is one.

5. Let A be an $n \times n$ nonsingular band matrix that satisfies the condition $a_{ij} = 0$ if $|i - j| > r$, where r is small, and let Gaussian elimination *with column pivoting* be used to solve $A\boldsymbol{x} = \boldsymbol{b}$. Identify all the coefficients of the intermediate equations that can become nonzero. Hence deduce that the total number of additions and multiplications of the complete calculation can be bounded by a constant multiple of nr^2.

6. Let a_1, a_2 and a_3 denote the columns of the matrix

$$A = \begin{bmatrix} 6 & 6 & 1 \\ 3 & 6 & 1 \\ 2 & 1 & 1 \end{bmatrix}.$$

Apply the Gram–Schmidt procedure to A, which generates orthonormal vectors $\boldsymbol{q}_1, \boldsymbol{q}_2$ and $\boldsymbol{q}_3$. Note that this calculation provides real numbers r_{jk} such that $\boldsymbol{a}_k = \sum_{j=1}^{k} r_{jk}\boldsymbol{q}_j$, $k = 1, 2, 3$. Hence express A as the product $A = QR$, where Q and R are orthogonal and upper-triangular matrices respectively.

7. Calculate the QR factorization of the matrix of Exercise 6 by using three Givens rotations. Explain why the initial rotation can be any one of the three types $\Omega^{(1,2)}$, $\Omega^{(1,3)}$ and $\Omega^{(2,3)}$. Prove that the final factorization is independent of this initial choice in exact arithmetic, provided that we satisfy the condition that in each row of R the leading nonzero element is positive.

8. Let A be an $n \times n$ matrix, and for $i = 1, 2, \ldots, n$ let $k(i)$ be the number of zero elements in the i-th row of A that come before all nonzero elements in this row and before the diagonal element a_{ii}. Show that the QR factorization of A can be calculated by using at most $\frac{1}{2}n(n-1) - \sum k(i)$ Givens rotations. Hence show that, if A is an upper triangular matrix except that there are nonzero elements in its first column, i.e. $a_{ij} = 0$ when $2 \le j < i \le n$, then its QR factorization can be calculated by using only $2n - 3$ Givens rotations. [*Hint*: Your should find the order of the first $(n-2)$ rotations that brings your matrix to the form considered above.]

9. Calculate the QR factorization of the matrix of Exercise 6 by using two Householder reflections. Show that, if this technique is used to generate the QR factorization of a general $n \times n$ matrix A, then the computation can be organised so that the total number of additions and multiplications is bounded above by a constant multiple of n^3.

10. Let

$$A = \begin{bmatrix} 3 & 4 & 7 & -2 \\ 5 & 4 & 9 & 3 \\ 1 & -1 & 0 & 3 \\ 1 & -1 & 0 & 0 \end{bmatrix}, \qquad \boldsymbol{b} = \begin{bmatrix} 11 \\ 29 \\ 16 \\ 10 \end{bmatrix}.$$

Calculate the QR factorization of A by using Householder reflections. In this case A is singular and you should choose Q so that the last row of R is zero. Hence identify all the least squares solutions of the inconsistent system $A\boldsymbol{x} = \boldsymbol{b}$, where we require $\boldsymbol{x}$ to minimize $\|A\boldsymbol{x} - \boldsymbol{b}\|_2$. Verify that all the solutions give the same vector of residuals $A\boldsymbol{x} - \boldsymbol{b}$, and that this vector is orthogonal to the columns of A. There is no need to calculate the elements of Q explicitly.