B6b Variational Principles: Example Sheet 2 Easter term 2017

Corrections and suggestions should be emailed to `p.k.townsend@damtp.cam.ac.uk`.

1. Obtain the Euler-Lagrange equation for the function $x(t)$ that makes stationary the functional

$$F[x] = \int_{t_1}^{t_2} f(t, x(t), \dot{x}(t), \ddot{x}(t))\, dt$$

for fixed values of both $x(t)$ and $\dot{x}(t)$ at both $t = t_1$ and $t = t_2$.

Given the boundary conditions

$$x(1) = 1\,, \quad \dot{x}(1) = -2\,; \qquad x(2) = \frac{1}{4}\,, \quad \dot{x}(2) = -\frac{1}{4}\,,$$

find the function $x(t)$ that minimises the integral $\int_1^2 t^4[\ddot{x}(t)]^2 dt$. Why is this function a global minimum of the integral?

2. A simple closed curve in the x-y plane is specified in terms of an angular parameter θ by the functions $x(\theta)$ and $y(\theta)$ for $0 \le \theta < 2\pi$. The area enclosed by the curve is

$$A[x, y] = \frac{1}{2}\int_0^{2\pi} (xy' - yx')\, d\theta\,.$$

Use this expression, and the Lagrange multiplier method, to find the curve that maximises the enclosed area for fixed length.

3. Using the Lagrange multiplier method, write down the Euler-Lagrange equations associated to the problem of minimising the functional

$$I[\psi] = \int_{-\infty}^{+\infty} \left(\psi'^2 + x^2\psi^2\right) dx$$

subject to the normalization condition $\int \psi^2 dx = 1$. Given that $x\psi(x)^2 \to 0$ as $x \to \pm\infty$, show that

$$I[\psi] = 1 + \int_{-\infty}^{+\infty} (\psi' + x\psi)^2 dx\,,$$

and hence deduce that $I \ge 1$. Show that equality holds for a function ψ that you should give explicitly. Verify that it satisfies the Euler-Lagrange equation for an appropriate value of the Lagrange multiplier.

4. Let $\mathbf{x}(t) \in \mathbb{R}^3$ be a curve which is constrained to lie on the sphere $S^2 = \{\mathbf{x} : |\mathbf{x}| = 1\}$. Use the Lagrange multiplier function formalism to obtain the following Euler-Lagrange equation

$$\ddot{\mathbf{x}} + |\dot{\mathbf{x}}|^2\mathbf{x} = \mathbf{0}$$

for the problem of minimising $I[\mathbf{x}] = \int |\dot{\mathbf{x}}|^2 dt$ amongst curves satisfying the constraint $\mathbf{x}(t) \in S^2$. Show that the solutions of the Euler-Lagrange equation lie on a plane through the origin (i.e. that they are great circles.)

5. A particle of mass m is constrained to roll on the inside of a smooth upturned hemispherical bowl of radius a. The Lagrangian describing the motion is

$$L = \frac{1}{2}ma^2\dot{\theta}^2 + \frac{1}{2}ma^2(\sin^2\theta)\dot{\phi}^2 + mga\cos\theta\,,$$

where g is the acceleration due to gravity, and θ and ϕ are the usual spherical angles (with θ measured relative to the downward vertical). Find two constants of the motion.

Find the two momenta p_θ and p_ϕ and hence the particle's Hamiltonian. What do Hamilton's equations become in this case?

6. Obtain the Euler-Lagrange equations associated with the functionals

   (i) $I[u] = \int [\frac{1}{2}u_t^2 - F(u_x)]dxdt\,,$ (ii) $I[u] = \int [|\nabla u|^2 + e^{2u}]dxdy\ .$

7. Hamilton's Principle is applicable to the *relativistic* dynamics of a charged particle in an electromagnetic field. The appropriate choice of Lagrangian $L[\mathbf{x}(t), \dot{\mathbf{x}}(t), t]$ for a particle of rest-mass m and charge q in a given electric potential $\phi(t,\mathbf{x})$ and magnetic vector potential $\mathbf{A}(t,\mathbf{x})$ is

$$L = -mc^2\sqrt{1 - |\mathbf{v}|^2/c^2} - q\phi + q\mathbf{v}\cdot\mathbf{A}\,,$$

   where $\mathbf{v} = \dot{\mathbf{x}}(t)$. Verify that the Euler-Lagrange equations yield the equation of motion

$$\frac{d}{dt}(m_0\gamma\mathbf{v}) = q(\mathbf{E} + \mathbf{v}\times\mathbf{B})\,, \qquad \gamma = (1 - |\mathbf{v}|^2/c^2)^{-\frac{1}{2}}\,,$$

   where $\mathbf{E} = -\nabla\phi - \partial\mathbf{A}/\partial t$ (the electric field) and $\mathbf{B} = \nabla\times\mathbf{A}$ (the magnetic field).

8. The mass density $\rho(t,\mathbf{x})$ and velocity field $\mathbf{v}(t,\mathbf{x})$ of a compressible fluid are constrained by conservation of mass to satisfy the continuity equation

$$\dot{\rho} + \nabla\cdot(\rho\mathbf{v}) = 0\,. \qquad (*)$$

   Given that the energy density of the fluid is $u(\rho)$, the action (for inviscid irrotational flow) is

$$S[\rho,\mathbf{v},\phi] = \int dt \int d^3x \left\{\frac{1}{2}\rho|\mathbf{v}|^2 - u(\rho) + \phi\,[\dot{\rho} + \nabla\cdot(\rho\mathbf{v})]\right\}\,,$$

   where $\phi(t,\mathbf{x})$ is a Lagrange multiplier field imposing the continuity condition $(*)$. Find the Euler-Lagrange equations for this action. Show that they imply $\mathbf{v} = \nabla\phi$ (so ϕ is the velocity potential). Given that the fluid pressure $P(t,\mathbf{x})$ satisfies

$$\nabla P = \rho\nabla h(t,\mathbf{x})\,, \qquad h = u'(\rho)\,,$$

   deduce Euler's equation for inviscid irrotational flow:

$$\rho\,[\dot{\mathbf{v}} + (\mathbf{v}\cdot\nabla)\,\mathbf{v}] = -\nabla P\,.$$

9. If a curve between points A and B on the unit sphere can be parametrised by the polar angle θ then its length is given by the functional $L[\phi] = \int_A^B (1 + \phi'^2\sin^2\theta)^{\frac{1}{2}}d\theta$. Show that $\delta^2 L$ is positive.

   If the curve can be parametrised by the azimuthal angle ϕ then its length is given by the functional $\tilde{L}[\theta] = \int_A^B (\theta'^2 + \sin^2\theta)^{\frac{1}{2}}d\phi$. Why does your result for $L[\phi]$ not imply that $\delta^2\tilde{L}$ is positive?

10. For $F[y] = \int_\alpha^\beta (y'^2 + y^4)dx$ with $y(\alpha) = a$, $y(\beta) = b$, show that $\delta^2 F$ is strictly positive, and hence that any solution of the Euler-Lagrange equation is a local minimum of F. Write down the Euler-Lagrange equation and find its solution for the case $a = b = 0$. Why is this solution a global minimum of F?

11. A function $y(x)$ defined for $0 \le x \le 1$ is such that $y(0) = y(1) = 0$. Write down the Euler-Lagrange equation associated to the functional

$$F[y] = \int_0^1 \left(\frac{1}{2}y'^2 + g(y)\right)dx\,,$$

   where $g(y)$ is such that $g'(0) = 0$. Show that $y_0(x) = 0$ is a solution. Given that the Euler-Lagrange equation is satisfied, find $\delta^2 F$ and determine the range of values of $g''(0)$ for which it is positive. [*This includes a range of negative values of* $g''(0)$.]