Complex Methods: Example Sheet 1

Part IB, Lent Term 2017
Dr R. E. Hunt

Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk. Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Cauchy–Riemann equations

1. (i) Where, if anywhere, in the complex plane are the following functions differentiable, and where are they analytic?

$$\operatorname{Im} z; \qquad |z|^2; \qquad \operatorname{sech} z.$$

(ii) Let $f(z) = z^5/|z|^4$, $z \neq 0$, $f(0) = 0$. Show that the real and imaginary parts of f satisfy the Cauchy–Riemann equations at $z = 0$, but that f is not differentiable at $z = 0$.

2. Find, as functions of z, complex analytic functions $f(z)$ whose real parts are the following:

$$\text{(i)}\ x \qquad \text{(ii)}\ xy \qquad \text{(iii)}\ \sin x \cosh y$$

$$\text{(iv)}\ \log(x^2+y^2) \qquad \text{(v)}\ \frac{y}{(x+1)^2+y^2} \qquad \text{(vi)}\ \tan^{-1}\left(\frac{2xy}{x^2-y^2}\right)$$

Deduce that the above functions are harmonic on appropriately-chosen domains, which you should specify.

∗ **3.** By considering $w(z) = (i+z)/(i-z)$, show that $\phi(x,y) = \tan^{-1}\dfrac{2x}{x^2+y^2-1}$ is harmonic.

4. Verify that the function $\phi(x,y) = e^x(x\cos y - y\sin y)$ is harmonic. Find its harmonic conjugate and, by considering $\nabla\phi$ or otherwise, determine the family of curves orthogonal to $\phi(x,y) = c$ for a given constant c.

Find an analytic function $f(z)$ such that $\operatorname{Re} f = \phi$. Can the expression $f(z) = \phi(z,0)$ be used to determine $f(z)$ in general?

Branches of multi-valued functions

5. Show how the principal branch of $\log z$ can be used to define a branch of z^i which is single-valued and analytic on the domain $\mathscr{D} = \mathbb{C} \setminus (-\infty, 0]$. Evaluate i^i for this branch.

Show, using polar coordinates, that the branch of z^i defined above maps $\mathscr{D}$ onto an annulus which is covered infinitely often.

How would your answers change, if at all, for a different branch?

6. Exhibit three different branches of the function $z^{3/2}$.

How many branch points does $[z(z+1)]^{1/3}$ have? Draw some possible branch cuts, both in the complex plane and on the Riemann sphere.

Repeat for $(z^2+1)^{1/2}$.

∗ Repeat also for $[z(z+1)(z+2)]^{1/3}$ and $[z(z+1)(z+2)(z+3)]^{1/2}$.

7. Let $f(z) = (z^2-1)^{1/2}$, and consider two different branches of the function $f(z)$:

$f_1(z)$: branch cut $[-1,1]$, with $f_1(x) = +\sqrt{x^2-1}$ for real $x > 1$;

$f_2(z)$: branch cut $(-\infty,-1] \cup [1,\infty)$, with $f_2(x) = +i\sqrt{1-x^2}$ for real $x \in (-1,1)$.

Find the limiting values of f_1 and f_2 above and below their respective branch cuts. Prove that f_1 is an odd function, i.e., $f_1(z) = -f_1(-z)$, and that f_2 is even.

Conformal mappings

8. How does the disc $|z-1|<1$ transform under the mapping $z \mapsto z^{-1}$?

Use the identity

$$\frac{z}{(z-1)^2} = \left(\frac{1}{1-z} - \frac{1}{2}\right)^2 - \frac{1}{4}$$

to show that the map $f(z) = z/(z-1)^2$ is a one-to-one conformal mapping of the disc $|z|<1$ onto the domain $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$.

9. Find conformal mappings f_i of $\mathscr{U}_i$ onto $\mathscr{V}_i$ for each of the following cases. If the mapping is a composition of several functions, provide a sketch for each step. $\mathscr{D}$ denotes the unit disc $|z|<1$.

(i) $\mathscr{U}_1$ is the angular sector $\{z : 0 < \arg z < \alpha\}$, $\mathscr{V}_1 = \{z : 0 < \operatorname{Im} z < 1\}$.

(ii) $\mathscr{U}_2 = \{z : \operatorname{Re} z < 0,\ -\frac{\pi}{2} < \operatorname{Im} z < \frac{\pi}{2}\}$, $\mathscr{V}_2 = \mathscr{D}$.

(iii) $\mathscr{U}_3 = \mathscr{D}$, $\mathscr{V}_3 = \mathscr{D} \setminus (-1, 0]$.

∗ (iv) $\mathscr{U}_4$ is the open region bounded between two circles $\{z : |z| < 1,\ |z+i| > \sqrt{2}\}$, $\mathscr{V}_4 = \mathscr{D}$.

Laplace's equation

10. Show that

$$\begin{aligned} g(z) = e^z \quad &\text{maps the strip } \mathscr{S} = \{z : 0 < \operatorname{Im} z < \pi\} \text{ onto the UHP } \{z : \operatorname{Im} z > 0\}, \\ h(z) = \sin z \quad &\text{maps the half-strip } \mathscr{H} = \{z : -\tfrac{\pi}{2} < \operatorname{Re} z < \tfrac{\pi}{2},\ \operatorname{Im} z > 0\} \text{ onto the UHP.} \end{aligned}$$

Find a conformal map $f\colon \mathscr{H} \to \mathscr{S}$. Hence find a function $\phi(x,y)$ which is harmonic on the half-strip $\mathscr{H}$ with the following limiting values on its boundary $\partial\mathscr{H}$:

$$\phi(x,y) = \begin{cases} 0 & \text{on } \partial\mathscr{H} \text{ in the LHP } (x<0), \\ 1 & \text{on } \partial\mathscr{H} \text{ in the RHP } (x>0). \end{cases}$$

Give ϕ as a function of x and y. Is there only one such function?

∗ **11.** Using conformal mapping(s), find a solution to Laplace's equation in the upper half-plane $\{(x,y) : y > 0\}$ with boundary conditions

$$\phi(x,0) = \begin{cases} 1 & x \in [-1,1], \\ 0 & \text{otherwise.} \end{cases}$$

[*Find a map f of the upper half-plane onto itself that makes the boundary conditions easier to deal with.*]

Series expansions

12. Find the first two non-vanishing coefficients in the series expansion about the origin of each of the following functions, assuming principal branches when there is a choice. You may make use of standard expansions for $\log(1+z)$, etc.

(i) $z/\log(1+z)$ (ii) $(\cos z)^{1/2} - 1$ (iii) $\log(1+e^z)$ (iv) e^{e^z}

State the range of values of z for which each series converges.

How would your answers differ if you assumed branches different from the principal branch?