Complex Methods: Example Sheet 1

Part IB, Lent Term 2024

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Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

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Cauchy–Riemann equations

1. (i) Where, if anywhere, in the complex plane are the following functions differentiable, and where are they analytic?

$$f(z) = \mathsf{Im}(z); \qquad f(z) = |z|^2; \qquad f(z) = \mathrm{sech} z := \frac{1}{\cosh z}.$$

(ii) Let $f(z) = z^5/|z|^4$, $z \neq 0$, $f(0) = 0$. Show that the real and imaginary parts of f satisfy the Cauchy–Riemann equations at $z = 0$, but that f is not differentiable at $z = 0$.

(iii) Given smooth functions $u(x, y)$, $v(x, y)$ we may define formally a function $g(z, \bar{z})$ by

$$g(z, \bar{z}) = u\big(\tfrac{1}{2}(z + \bar{z}), -\tfrac{1}{2}i(z - \bar{z})\big) + iv\big(\tfrac{1}{2}(z + \bar{z}), -\tfrac{1}{2}i(z - \bar{z})\big).$$

Using the chain rule and the Cauchy–Riemann equations, show that g is differentiable as a function of z if and only if $\partial g/\partial \bar{z} = 0$, and discuss how a dependence on $\bar{z}$ results in the nondifferentiability of f.

2. By calculation or inspection, find (as functions of z) complex analytic functions $f(z)$ whose real parts are the following:

(i) xy (ii) $\sin x \cosh y$ (iii) $\log(x^2 + y^2)$

(iv) $e^{y^2 - x^2} \cos 2xy$ (v) $\dfrac{y}{(x + 1)^2 + y^2}$ (vi) $\tan^{-1}\left(\dfrac{2xy}{x^2 - y^2}\right)$

Deduce that the above functions are harmonic on appropriately-chosen domains, which you should specify.

∗ **3.** By considering $w(z) = (\mathrm{i} + z)/(\mathrm{i} - z)$, show that $\phi(x, y) = \tan^{-1}\left(\dfrac{2x}{x^2 + y^2 - 1}\right)$ is harmonic.

4. Verify that the function $\phi(x, y) = e^x(x \cos y - y \sin y)$ is harmonic. Find its harmonic conjugate and, by considering $\nabla\phi$ or otherwise, determine the family of curves orthogonal to the family of curves $\phi(x, y) = \text{const}$.

Find an analytic function $f(z)$ such that $\mathsf{Re}(f) = \phi$. Can the expression $f(z) = \phi(z, 0)$ be used to determine $f(z)$ in general?

Branches of multi-valued functions

5. Show how the principal branch of $\log z$ can be used to define a branch of z^{i} which is single-valued and analytic on the domain $\mathcal{D} = \mathbb{C} \setminus (-\infty, 0]$. Evaluate i^{i} for this branch.

Show, using polar coordinates, that the branch of z^{i} defined above maps $\mathcal{D}$ onto an annulus which is covered infinitely often.

How would your answers change, if at all, for a different branch?

6. Exhibit three different branches of the function $z^{3/2}$ using at least two different branch cuts. Is it true that for $z, w \in \mathbb{C}$, $(zw)^{3/2} = z^{3/2}w^{3/2}$?

How many branch points does $[z(z+1)]^{1/3}$ have? Draw some different choices of possible branch cuts, both in the complex plane and on the Riemann sphere.

Repeat for $(z^2+1)^{1/2}$.

$*$ Repeat also for $[z(z+1)(z+2)]^{1/3}$ and $[z(z+1)(z+2)(z+3)]^{1/2}$.

7. Let $f(z) = (z^2-1)^{1/2}$, and consider two different branches of the function $f(z)$:

$$f_1(z) : \text{branch cut } (-\infty, -1] \cup [1, \infty), \text{ with } f_1(x) = -\mathrm{i}\sqrt{1-x^2} \text{ for real } x \in (-1, 1);$$
$$f_2(z) : \text{branch cut } [-1, 1], \text{ with } f_2(x) = +\sqrt{x^2-1} \text{ for real } x > 1.$$

Find the limiting values of f_1 and f_2 above and below their respective branch cuts. Prove that f_1 is an even function, i.e., $f_1(z) = f_1(-z)$, and that f_2 is odd.

Conformal mappings

8. How does the disc $|z-1| < 1$ transform under the mapping $z \mapsto z^{-1}$?

Use the identity

$$\frac{z}{(z-1)^2} = \left(\frac{1}{1-z} - \frac{1}{2}\right)^2 - \frac{1}{4}$$

to show that the map $f(z) = z/(z-1)^2$ is a one-to-one conformal mapping of the disc $|z| < 1$ onto the domain $\mathbb{C} \setminus (-\infty, -\frac{1}{4}]$.

9. Consider the complex plane partitioned into eight open regions using as boundaries the real axis, the imaginary axis and the unit circle. Show that the map $z \mapsto (z-1)/(z+1)$ permutes these regions, and find the permutation.

What is the action of $z \mapsto (z-\mathrm{i})/(z+\mathrm{i})$ on these regions?

10. Find conformal mappings f_i of $\mathcal{U}_i$ onto $\mathcal{V}_i$ for each of the following cases. If the mapping is a composition of several functions, provide a sketch for each step. $\mathcal{D}$ denotes the unit disc $|z| < 1$.

(i) $\mathcal{U}_1$ is the angular sector $\{z : 0 < \arg z < \alpha\}$, $\mathcal{V}_1 = \{z : 0 < \mathsf{Im}(z) < 1\}$.

(ii) $\mathcal{U}_2 = \{z : \mathsf{Re}(z) < 0,\ -\frac{\pi}{2} < \mathsf{Im}(z) < \frac{\pi}{2}\}$, $\mathcal{V}_2$ is the quadrant $\{z : \mathsf{Re}(z) > 0,\ \mathsf{Im}(z) > 0\}$.

(iii) $\mathcal{U}_3 = \mathcal{D}$, $\mathcal{V}_3 = \mathcal{D} \setminus (-1, 0]$.

$*$ (iv) $\mathcal{U}_4$ is the open region bounded between two circles $\{z : |z| < 1,\ |z+\mathrm{i}| > \sqrt{2}\}$, $\mathcal{V}_4 = \mathcal{D}$.

Laplace's equation

11. Consider the half-strip $\mathcal{H} = \{z : -\frac{\pi}{2} < \mathsf{Re}(z) < \frac{\pi}{2},\ \mathsf{Im}(z) > 0\}$, the strip $\mathcal{S} = \{z : 0 < \mathsf{Im}(z) < \pi\}$, and the upper half-plane (UHP) $\{z : \mathsf{Im}(z) > 0\}$. Show that $g(z) = e^z$ maps $\mathcal{S}$ onto the UHP and that $h(z) = \sin z$ maps $\mathcal{H}$ onto the UHP.

Find a conformal map $f\colon \mathcal{H} \to \mathcal{S}$. Hence find a function $\phi(x, y)$ which is harmonic on the half-strip $\mathcal{H}$ with the following limiting values on its boundary $\partial\mathcal{H}$:

$$\phi(x, y) = \begin{cases} 0 & \text{on } \partial\mathcal{H} \text{ in the LHP } (x < 0), \\ 1 & \text{on } \partial\mathcal{H} \text{ in the RHP } (x > 0). \end{cases}$$

Give ϕ as a function of x and y. Is there only one such function?

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$*$ **12.** Using conformal mapping(s), find a solution to Laplace's equation in the upper half-plane $\{(x, y) : y > 0\}$ with boundary conditions

$$\phi(x, 0) = \begin{cases} 1 & x \in [-1, 1], \\ 0 & \text{otherwise.} \end{cases}$$

[*Find a map f of the upper half-plane onto itself that makes the boundary conditions easier to deal with.*]

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