Complex Methods: Example Sheet 2

Part IB, Lent Term 2017
Dr R. E. Hunt

Comments on or corrections to this example sheet are very welcome and may be sent to reh10@cam.ac.uk.

Starred questions are useful, but optional: they should not be attempted at the expense of other questions.

Singularities and Laurent series

1. Let a, b be complex constants with $0 < |a| < |b|$. Use partial fractions to find the Laurent expansions of $1/\{(z-a)(z-b)\}$ about $z = 0$ in each of the regions $|z| < |a|$, $|a| < |z| < |b|$ and $|z| > |b|$.

2. Find the first three terms of the Laurent expansion of $f(z) = \operatorname{cosec}^2 z$ valid for $0 < |z| < \pi$.

$*$ Show that the function $g(z) = f(z) - z^{-2} - (z+\pi)^{-2} - (z-\pi)^{-2}$ has only removable singularities in $|z| < 2\pi$. Explain how to remove them to obtain a function $G(z)$ analytic in that region. Find a Taylor Series for $G(z)$ about the origin and explain why it must be convergent in $|z| < 2\pi$. Hence, or otherwise, find the three non-zero central terms of the Laurent expansion of $f(z)$ valid for $\pi < |z| < 2\pi$.

3. Write down the location and type of each of the singularities of the following functions:

(i) $\dfrac{1}{z^3(z-1)^2}$ (ii) $\tan z$ (iii) $z \coth z$ (iv) $\dfrac{e^z - e}{(1-z)^3}$

(v) $\exp(\tan z)$ (vi) $\sinh \dfrac{z}{z^2-1}$ (vii) $\log(1+e^z)$ (viii) $\tan(z^{-1})$.

Integration and residues

4. Evaluate $\int z \, \mathrm{d}z$ along the straight line from -1 to $+1$, and along the semicircular contour in the upper half-plane between the same two points; and evaluate $\oint_\gamma \bar{z} \, \mathrm{d}z$ when γ is the circle $|z| = 1$, and when γ is the circle $|z-1| = 1$.

5. (i) Show that if $f(z)$ is analytic, then the residue of $f(z)/(z-z_0)$ at $z = z_0$ is $f(z_0)$.

(ii) Show that if $1/f(z)$ has a simple pole at $z = z_0$, then its residue at $z = z_0$ is $1/f'(z_0)$.

(iii) Show that if $h(z)$ has a simple zero at $z = z_0$ and $g(z)$ is analytic and non-zero, the residue of $g(z)/h(z)$ at $z = z_0$ is $g(z_0)/h'(z_0)$.

(iv) Prove the formula for the residue of a function $f(z)$ that has a pole of order N at $z = z_0$:

$$\lim_{z \to z_0} \left\{ \frac{1}{(N-1)!} \frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}} \big((z-z_0)^N f(z) \big) \right\}.$$

6. Evaluate, using Cauchy's theorem or the residue theorem,

(i) $\displaystyle\oint_{\gamma_1} \frac{\mathrm{d}z}{1+z^2}$ (ii) $\displaystyle\oint_{\gamma_2} \frac{\mathrm{d}z}{1+z^2}$ (iii) $\displaystyle\oint_{\gamma_3} \frac{e^z \cos z \, \mathrm{d}z}{(1+z^2)\sin z}$ $*$ (iv) $\displaystyle\oint_{\gamma_4} \frac{z^3 e^{1/z} \, \mathrm{d}z}{1+z}$

where γ_1 is the elliptical contour $(\operatorname{Re} z)^2 + 4(\operatorname{Im} z)^2 = 1$, γ_2 is the circle $|z| = \sqrt{2}$, γ_3 is the circle $|z - (2+i)| = \sqrt{2}$ and γ_4 is the circle $|z| = 2$, all traversed anti-clockwise.

7. By integrating the function $z^n(z-a)^{-1}(z-a^{-1})^{-1}$ around the unit circle, evaluate

$$\int_0^{2\pi} \frac{\cos n\theta}{1-2a\cos\theta+a^2}\,\mathrm{d}\theta$$

where a is real, $a>1$, and n is a non-negative integer.

The calculus of residues

8. Evaluate $\displaystyle\int_{-\infty}^{\infty} \frac{\mathrm{d}x}{1+x+x^2}$.

∗ Also evaluate $\displaystyle\lim_{R\to\infty}\int_{-R}^{R} \frac{x\,\mathrm{d}x}{1+x+x^2}$. Why is the limit here rather than just $\displaystyle\int_{-\infty}^{\infty}$?

9. By integrating around a keyhole contour, show that

$$\int_0^{\infty} \frac{x^{a-1}\,\mathrm{d}x}{1+x} = \frac{\pi}{\sin\pi a} \qquad (0<a<1).$$

Explain why the given restrictions on the value of a are necessary.

∗ **10.** By integrating around a contour involving the real axis and the line $z = re^{2\pi i/n}$, evaluate

$$\int_0^{\infty} \frac{\mathrm{d}x}{1+x^n} \qquad (n \geqslant 2).$$

Check (by change of variable) that your answer agrees with that of the previous question.

11. Establish the following:

$$\text{(i)}\ \int_0^{\infty} \frac{\cos x}{(1+x^2)^3}\,\mathrm{d}x = \frac{7\pi}{16e} \qquad \text{(ii)}\ \int_0^{\infty} \frac{\sin^2 x}{x^2}\,\mathrm{d}x = \frac{\pi}{2} \qquad \text{(iii)}\ \int_0^{\infty} \frac{\log x}{1+x^2}\,\mathrm{d}x = 0.$$

[*For part (iii), integrate* $(\log z)^2/(1+z^2)$ *around a keyhole, or* $\log z/(1+z^2)$ *along the real axis (or both). What goes wrong if you integrate* $\log z/(1+z^2)$ *around a keyhole?*]

∗ **12.** Let $P(z)$ be a non-constant polynomial. Consider the contour integral

$$I = \oint_\gamma \frac{P'(z)}{P(z)}\,\mathrm{d}z.$$

Show that, if γ is a contour that encloses no zeros of P, then $I=0$.

Evaluate the limit of I as $R\to\infty$, where γ is the circle $|z|=R$, and deduce that P has at least one zero in the complex plane.

13. By considering the integral of $f(z) = \cot z/(z^2+\pi^2a^2)$ around a suitable large contour, prove that, provided ia is not an integer,

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = \frac{\pi}{a}\coth\pi a.$$

By considering a similar integral prove also that, if a is not an integer,

$$\sum_{n=-\infty}^{\infty} \frac{1}{(n+a)^2} = \frac{\pi^2}{\sin^2\pi a}.$$

Find an expression for $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2+a^2}$ and take the limit as $a\to 0$ to deduce the value of $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2}$.