Course C10—No. B6

METHODS — OLD EXAMPLES I (2015)
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Fourier series

1. *Fourier coefficients (full-range series).* For the periodic function $f(\theta) = (\theta^2 - \pi^2)^2$ on the interval $-\pi \le \theta < \pi$, show that it has the Fourier series

$$f(\theta) = \frac{8\pi^4}{15} + 24 \sum_{n \neq 0} \frac{(-1)^{n+1}}{n^4} \mathrm{e}^{\mathrm{i}n\theta} \,.$$

[Remark: if you're happy that you can do the integrals you might like to save time by using www.integrals.com to evaluate them.] Sketch the function $f(\theta)$ and comment on its differentiability and the order of the terms in its Fourier series as $n \to \infty$.

2. *Fourier coefficients (half-range series).* Suppose that $f(\theta) = \theta^2$ for $0 \le \theta \le \pi$. Express $f(\theta)$ as (a) a Fourier sine series, and (b) a cosine series, each having period 2π. Sketch the functions represented by (a) and (b) in the range -6π to 6π. If the series (a) and (b) are differentiated term-by-term, how are the answers related (if at all) to the Fourier series for $g(\theta) = 2\theta$ and $h(\theta) = 2|\theta|$ each in the range $(-\pi, \pi)$?

3. *Series summation.* Find the (complex) Fourier series of $f(\theta) = e^\theta$ for $\theta \in [-\pi, \pi)$. Deduce that

$$\sum_{n=1}^{\infty} \frac{1}{1+n^2} = \frac{1}{2}(\pi \coth \pi - 1) \quad .$$

4. *Parseval's identity and a low pass filter.* (i) Given that a function $f(t)$ defined over the interval $(-T, T)$ has the Fourier series

$$f(t) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi t}{T}\right) + b_n \sin\left(\frac{n\pi t}{T}\right) \right] , \qquad \text{show directly that} \qquad \frac{1}{T}\int_{-T}^{T} [f(t)]^2 dt = \frac{1}{2}a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) \,,$$

where you may assume $f(t)$ is such that this series is convergent.
(ii) A unit amplitude square wave of period $2T$ is given by $f(t) = 1$ for $0 < t < T$ and $f(t) = -1$, for $-T < t < 0$. Suppose this is the input for a system which permits angular frequencies less than $\frac{9}{2}\pi T^{-1}$ to be perfectly transmitted and frequencies greater than $\frac{9}{2}\pi T^{-1}$ to be perfectly absorbed. Calculate the form of the output. The power is proportional to the mean value of $f^2(t)$; what fraction of the incident power is transmitted?

5. *Discontinuities and the Wilbraham-Gibbs phenomenon**. (i) Suppose that f is a square wave given by

$$f(\theta) = \begin{cases} 1 & 0 < \theta < \pi \\ 0 & \pi < \theta < 2\pi \,. \end{cases} \qquad \text{Sketch } f \text{ and show that} \quad f(\theta) = \frac{1}{2} + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(2k-1)\theta}{2k-1} \,.$$

(ii) Now define the partial sum of this series as $\quad S_n(\theta) = \dfrac{1}{2} + \dfrac{2}{\pi} \displaystyle\sum_{k=1}^{n} \frac{\sin(2k-1)\theta}{2k-1} \quad ,$

and find the following expression $\quad S_n(\theta) = \dfrac{1}{2} + \dfrac{1}{\pi} \displaystyle\int_0^\theta \frac{\sin 2nt}{\sin t} \mathrm{d}t$. [Hint: consider $S_n'(\theta)$ for the two forms.]

(iii) Deduce that $S_n(\theta)$ has extrema at $\theta = m\pi/2n$, $n = 1, 2, \ldots 2n-1, 2n+1, \ldots$, (*i.e.*, all integer m except even multiples of n) and that the height of the first maximum for large n is approximately

$$S_n\left(\frac{\pi}{2n}\right) = \frac{1}{2} + \frac{1}{\pi} \int_0^\pi \frac{\sin u}{u} \, \mathrm{d}u \; (\simeq \; 1{\cdot}089) \,.)$$

Comment on the accuracy of Fourier series at discontinuities. (This question takes you through some important steps which are used in the proof of Fourier's theorem – refer, for example, to chapter 14 of Jeffreys & Jeffreys.)

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Sturm-Liouville theory

6. *Eigenfunctions and eigenvalues*. In the boundary value problem

$$y'' + \lambda y = 0; \; y(0) = 0, \quad y(1) + y'(1) = 0,$$

show that the eigenvalue λ can take infinitely many values $\lambda_1 < \lambda_2 < \lambda_3 ...$ Indicate roughly the behaviour of λ_n as $n \to \infty$.

7. *Recasting in Sturm-Liouville form*. Express the following equations in Sturm-Liouville form:

$$\text{(i)} \;\; (1-x^2)y'' - 2xy' + n(n+1)y = 0\,, \qquad \text{(ii)} \;\; xy'' + (b-x)y' - ay = 0\,,$$

where n, a, and b are constants.

(iii) Find the eigenvalues and eigenfunctions for

$$y'' + 4y' + (4+\lambda)y = 0, \; y(0) = y(1) = 0.$$

What is the orthogonality relation for these eigenfunctions?

8. *Bessel's equation*. (i) Show that the eigenvalues of the Sturm-Liouville problem

$$\frac{d}{dx}(x\frac{du}{dx}) + \lambda x u = 0\,, \qquad 0 < x < 1\,,$$

with $u(x)$ bounded as $x \to 0$ and $u(1) = 0$, are $\lambda = j_n^2$ $(n = 1, 2, ...)$, where the j_n are the zeros of the Bessel function $J_0(z)$, arranged in ascending order. [Recall: Bessel's equation of order zero is $\frac{d}{dz}(z\frac{dy}{dz}) + zy = 0\,, (z > 0)$, which you may assume has one solution $J_0(z)$ which is regular at $z = 0$ and a second linearly independent solution $Y_0(z)$ which is singular at $z = 0$.]

(ii) Using integration by parts on the differential equations for $J_0(\alpha x)$ and $J_0(\beta x)$, show that

$$\int_0^1 J_0(\alpha x) J_0(\beta x) x dx = \frac{\beta J_0(\alpha) J_0^{'}(\beta) - \alpha J_0(\beta) J_0^{'}(\alpha)}{\alpha^2 - \beta^2} \quad (\beta \neq \alpha)$$

$$\int_0^1 J_0(j_n x) J_0(j_m x) x dx = 0\,, \quad (n \neq m)\,, \qquad \int_0^1 [J_0(j_n x)]^2 x dx = \frac{1}{2}[J_0^{'}(j_n)]^2. \quad [\textit{Hint:} \text{ Consider } \beta = j_n + \epsilon \text{ as } \epsilon \to 0.]$$

(iii) Assume that the inhomogeneous equation

$$\frac{d}{dx}(x\frac{du}{dx}) + \tilde{\lambda} x u = x f(x),$$

where $\tilde{\lambda}$ is not an eigenvalue, has a unique solution such that $u(x)$ is bounded as $x \to 0$ and $u(1) = 0$. Assuming also that $f(x)$ satisfies the same boundary conditions as u and the completeness of the eigenfunctions $J_0(j_n x)$, obtain the eigenfunction expansion of u.

9. *Higher order self-adjoint form**. Consider the fourth-order differential operator

$$\mathcal{L} = \sum_{r=0}^{4} p_r(x) \frac{d^r}{dx^r},$$

where the $p_r(x)$ are real functions, with BCs $y(0) = y(1) = y'(0) = y'(1) = 0$. Show that $\mathcal{L}$ is self-adjoint if and only if $p_3 = 2p_4'$, $p_1 = p_2' - p'''_4$.

Considering a specific example, show that the boundary value problem

$$-y'''' + \lambda y = 0; \; y(0) = y(1) = y'(0) = y'(1) = 0$$

has infinitely many eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 ...$. Indicate roughly the behaviour of λ_n as $n \to \infty$.

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