METHODS — EXAMPLES II (2015)
2016 VERSION AT `http://www.damtp.cam.ac.uk/user/dbs26/`

The one-dimensional wave equation

1. *Modes on a string.* A uniform string of line density μ and tension T undergoes small transverse vibrations of amplitude $y(x,t)$. The string is fixed at $x=0$ and $x=\ell$, and satisfies the initial conditions

$$y(x,0)=0\,, \qquad y_t(x,0)=\frac{4V}{\ell^2}x(\ell-x)\,, \quad \text{for} \quad 0<x<\ell\,,$$

where $y_t \equiv \partial y/\partial t$. Using the fact that $y(x,t)$ is a solution of the wave equation, find the amplitudes of the normal modes and deduce the kinetic and potential energies of the string at time t. By comparison with the initial energy of the string show that

$$\sum_{\text{n odd}} \frac{1}{n^6}=\frac{\pi^6}{960}\,.$$

2. *Damped string motion.* (i) A uniform stretched string of length ℓ, density per unit length μ and tension $T=\mu c^2$ is fixed at both ends. The motion of the string is resisted by the surrounding medium, the resistive force on unit length being $-2k\mu y_t$, where y is the transverse displacement and the constant $k=\pi c/\ell$. Show that the equation of motion of the string is

$$c^2\frac{\partial^2 y}{\partial x^2}=\frac{\partial^2 y}{\partial t^2}+2k\frac{\partial y}{\partial t}\,,$$

and find $y(x,t)$ given that $y(x,0)=A\sin(\pi x/\ell)$ and $y_t(x,0)=0$.

(ii) If an extra transverse force $F_0\sin(\pi x/\ell)\cos(\pi ct/\ell)$ per unit length acts on the string, find the resulting forced oscillation. [*Hint:* You may find it useful to guess a particular solution to combine with the general homogeneous solution that you probably derived in (i).]

3. *Wave reflection and transmission.* A string of uniform density is stretched along the x-axis under tension T and undergoes small transverse oscillations in the (x,y) plane so that its displacement $y(x,t)$ satisfies

$$\frac{\partial^2 y}{\partial t^2}=c^2\frac{\partial^2 y}{\partial x^2}\,, \qquad (*)$$

where c is a constant.

(i) Show that if a mass M is fixed to the string at $x=0$ then its equation of motion can be written

$$M\left.\frac{\partial^2 y}{\partial t^2}\right|_{x=0}=T\left[\frac{\partial y}{\partial x}\right]_{x=0_-}^{x=0_+}\,.$$

(ii) Suppose that a wave $\exp[i\omega(t-x/c)]$ is incident from $x=-\infty$. Obtain the amplitudes and phases of the reflected and transmitted waves, and comment on their values when $\lambda=M\omega c/T$ is large or small.

4. *Impulsive force on a string.* The displacement $y(x,t)$ of a uniform string stretched between the points $x=0,\ell$ satisfies the wave equation $(*)$ given above with the boundary conditions, $y(0,t)=y(\ell,t)=0$. For $t<0$ the string oscillates in its fundamental mode and $y(x,0)=0$. A musician strikes the midpoint of the string impulsively at time $t=0$ so that the change in $\partial y/\partial t$ at $t=0$ is $\lambda\delta(x-\frac{1}{2}\ell)$. Find $y(x,t)$ for $t>0$.

Laplace's equation

5. *Cartesian coordinates.* Show that the solution of $\nabla^2\phi=0$ in the region $0<x<a, 0<y<b, 0<z<c$, with $\phi=1$ on the surface $z=0$ and $\phi=0$ on all the other surfaces is

$$\phi=\frac{16}{\pi^2}\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{\sinh[\ell(c-z)]\sin[(2p+1)\pi x/a]\sin[(2q+1)\pi y/b]}{(2p+1)(2q+1)\sinh c\ell}\,,$$

where $\ell^2=(2p+1)^2\pi^2/a^2+(2q+1)^2\pi^2/b^2$. [*Hint:* You may find it useful to use the above form of the z-dependent part of the solution from the outset.] Discuss the behaviour of the solutions as $c\to\infty$.

6. _Plane polar coordinates._ The potential ϕ satisfies Laplace's equation in the unit circle $r < 1$ with boundary condition

$$\phi(r=1,\theta) = \begin{cases} \pi/2\,, & 0 \le \theta < \pi\,. \\ -\pi/2\,, & \pi \le \theta < 2\pi\,. \end{cases}$$

Using the method of separation of variables show that

$$\phi(r,\theta) = 2 \sum_{n\ odd} \frac{r^n \sin n\theta}{n}\,.$$

Sum the series using the substitution $z = re^{i\theta}$. [Your solution can then be interpreted geometrically in terms of the angle between the lines to the two points on the boundary where the data jumps.]

Legendre polynomials

7. _Eigenfunction derivatives._ If y_m and y_n are real eigenfunctions of the Sturm-Liouville equation

$$\frac{d}{dx}(p(x)\frac{dy}{dx}) + (\lambda - q(x))y = 0\,,\ (a < x < b)\,,\ \text{ satisfying the normalisation condition } \int_a^b y_m^2 dx = \int_a^b y_n^2 dx = 1\,,$$

show that (under suitable boundary conditions)

$$\int_a^b (py_m^{'}\, y_n^{'} + qy_m y_n)dx = \lambda_m \delta_{mn} \quad \text{(no summation)}.$$

With P_n a Legendre polynomial, use this result to evaluate $\int_{-1}^{1} (1-x^2)P_m^{'}(x)P_n^{'}(x)dx\,.$

8. _Legendre polynomials and multipole moments._ Show that $1/|\mathbf{r} - \mathbf{k}|$ obeys Laplace's equation in three dimensions whenever $\mathbf{r} \neq \mathbf{k}$. Taking $\mathbf{k}$ to be a unit vector in the z-direction, show that

$$P_\ell(x) = \frac{1}{\ell!}\frac{d^\ell}{dr^\ell}\frac{1}{\sqrt{1-2r\cos\theta + r^2}}\bigg|_{r=0}$$

by expanding this solution of Laplace's equation in the region $r < 1$. Use the integral

$$\int_{-1}^{1} \frac{1}{1-2rx+r^2}\,dx$$

to show that the Legendre polynomials obey the normalization condition $\int_{-1}^{1} P_\ell(x)^2 dx = 2/(2\ell+1)$. Show also that $P_{\ell+1}'(x) - P_{\ell-1}'(x) = (2\ell+1)P_\ell(x)$.

9. _Spherical polar coordinates._ You've just shown that the electrostatic potential in a charge-free region satisfies Laplace's equation. Find the potential inside a spherical region bounded by two (conducting) hemispheres upon which the potential takes the values $\pm V$ respectively. [*Hint:* Note that $\int_{-1}^{1} P_m(x)P_n(x)dx = \frac{2}{2n+1}\delta_{mn}$.]

The heat equation

10. _Diffusion in a disc & Bessel functions._ Consider the unit disc, with initial temperature distribution $\psi_0(r,\theta)$. Require the boundary of the disc to be held at (wlog) zero temperature $\psi(1,\theta,t) = 0$ for all $t > 0$. By assuming that the temperature satisfies the diffusion equation in the disc (with unit diffusion coefficient) show that the solution is

$$\psi = \sum_{n=-\infty}^{\infty}\sum_{k=1}^{\infty} c_{nk} J_n(j_{nk} r)\exp[in\theta - j_{nk}^2 t],$$

where j_{nk} is the k^{th} smallest (positive) zero of the n^{th} order Bessel function of the first kind, (i.e. $J_n(j_{nk}) = 0$) and present an appropriate expression for c_{nk}, showing explicitly that

$$\int_0^1 J_n(j_{nk}r)J_n(j_{nl}r)rdr = \frac{\delta_{kl}[J_n'(j_{nk})]^2}{2} = \frac{\delta_{kl}J_{n+1}^2(j_{nk})}{2}.$$

Suppose now that the initial temperature $\psi_0(r,\theta) = \Psi_0$ is constant. Show that the only non-zero coefficients have $n = 0$, and are equal to

$$c_{0k} = \frac{2\Psi_0}{j_{0k}J_1(j_{0k})}.$$

What is the spatial structure of the temperature distribution as $t \to \infty$?
[*The recursion relations* $[z^{-\nu}J_\nu(z)]' = -z^{-\nu}J_{\nu+1}(z)$ *and* $[z^{\nu+1}J_{\nu+1}(z)]' = z^{\nu+1}J_\nu(z)$ *may be useful, as is the fact that Q8 of the first example sheet can be generalized straightforwardly to* J_n *for arbitrary positive integers* n.]