METHODS — EXAMPLES III

Green's functions

1. *Initial value problem.* The reading $\theta(t)$ of an ammeter satisfies

$$\ddot{\theta} + 2p\dot{\theta} + (p^2 + q^2)\theta = f(t)\,,$$

where p, q are constants with $p > 0$. The ammeter is set so that θ and $\dot{\theta}$ are zero when $t = 0$. Assuming $q \neq 0$, show by constructing the Green's function that

$$\theta(t) = \frac{1}{q}\int_0^t e^{-p(t-\tau)} \sin[q(t-\tau)] f(\tau) d\tau\,.$$

Derive the same result using Fourier transforms, showing that the transfer function for this system is

$$\tilde{R}(\omega) = \frac{1}{2qi}\left[\frac{1}{(i\omega + p - qi)} - \frac{1}{(i\omega + p + qi)}\right].$$

2. *Boundary value problem.* Obtain the Green's function $G(x, \xi)$ satisfying

$$\frac{d^2G}{dx^2} - \lambda^2 G = \delta(x - \xi)\,, \qquad 0 \leq x \leq 1\,, \quad 0 \leq \xi \leq 1\,.$$

where λ is real, subject to the boundary conditions $G(0, \xi) = G(1, \xi) = 0$. Show that the solution to the equation

$$\frac{d^2y}{dx^2} - \lambda^2 y = f(x), \qquad \text{subject to the same boundary conditions is}$$

$$y = -\frac{1}{\lambda \sinh\lambda}\left\{\sinh\lambda x \int_x^1 f(\xi)\sinh\lambda(1-\xi)d\xi + \sinh\lambda(1-x)\int_0^x f(\xi)\sinh\lambda\xi d\xi\right\}.$$

3. *Finite asymptotics.* A linear differential operator is defined by

$$L_x y = -\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + y\,.$$

By writing $y = z/x$ or otherwise, find those solutions of $L_x y = 0$ which are either (a) bounded as $x \to 0$, or (b) bounded as $x \to \infty$. Find the Green's function $G(x, a)$ satisfying

$$L_x G(x, a) = \delta(x - a)\,,$$

and both conditions (a) and (b). Use $G(x, a)$ to solve (subject to conditions (a) and (b))

$$L_x y(x) = \begin{cases} 1, & \text{for } 0 \leq x \leq R\,, \\ 0, & \text{for } x > R\,. \end{cases}$$

Show that the solution has the form, for suitable constants A, B

$$y(x) = \begin{cases} 1 + Ax^{-1}\sinh x\,, & \text{for } 0 \leq x \leq R\,, \\ Bx^{-1}e^{-x}\,, & \text{for } x > R\,. \end{cases}$$

4. *Higher order initial value problem*.* Show that the Green's function for the initial value problem ($' \equiv \frac{d}{dt}$)

$$y'''' + k^2 y'' = f(t)\,, \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0\,,$$

$$\text{is given by} \qquad G(t, \tau) = \begin{cases} 0\,, & 0 < t < \tau\,, \\ k^{-2}(t-\tau) - k^{-3}\sin k(t-\tau)\,, & t > \tau\,. \end{cases}$$

Hence find the solution for $f(t) = e^{-t}$.
[*Hint:* To make the calculations easier, for $t > \tau$ write the general homogeneous solution as a function of $t - \tau$.]

The Dirac delta function

5. *Delta function properties*. The function $\phi(x)$ is monotone increasing in $[a,b]$ and has a (simple) zero at $x=c$ (i.e. $\phi'(c)\neq 0$) where $a<c<b$. Show that

$$\int_a^b f(x)\delta[\phi(x)]dx = \frac{f(c)}{|\phi'(c)|}\ .$$

Show that the same formula applies if $\phi(x)$ is monotone decreasing and hence derive a formula for general $\phi(x)$ provided the zeros are simple. Deduce that $\delta(at)=\delta(t)/|a|$ for $a\neq 0$. Also establish that

$$\int_{-\infty}^{+\infty} |x|\delta(x^2-a^2)dx = 1 \quad .$$

6. *Delta function derivative**. Show using polar coordinates that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x^2+y^2)\delta'(x^2+y^2-1)\delta(x^2-y^2)dx\,dy = f(1)-f'(1)$$

(where you may assume that $f(r^2)/r$ has a finite limit at $r=0$).

Fourier transforms

7. *Duality property of FTs*. (a) Use the duality property of FTs (i.e. that $FT(FT(f))(x)=2\pi f(-x)$) to derive the frequency shift property from the translation property. (b) Let $h(x)=f(x)g(x)$. Starting with the convolution theorem (that FT of a convolution is the product of individual FTs) show that $\tilde{h}(k)=\tilde{f}*\tilde{g}(k)/2\pi$.

8. *Functions with discontinuities*. Let $f(x)=e^{-x}$ for $0<x<\infty$, and $f(x)=0$ for $x<0$. Show that $\tilde{f}(k)=\frac{1-ik}{1+k^2}$. Show that the inverse Fourier transform of this Fourier transform $\tilde{f}(k)$ takes the value of $1/2$ at $x=0$. (This is a general property of Fourier transforms, analogously to Fourier series. Inversion for general x is really straightforward with Complex Methods.)

9. *Fourier transform of Gaussians*. By using differentiation and the shift property, calculate the Fourier transform of a Gaussian distribution with a peak at $\mu\neq 0$, i.e. $f(x)=\exp[-n^2(x-\mu)^2]$.
Now let $\mu=0$, and consider $\delta_n(x)=(n/\sqrt{\pi})f(x)$. Sketch $\delta_n(x)$ and $\tilde{\delta}_n(k)$ for small and large n. What is $\int_{-\infty}^{\infty}\delta_n(x)dx$? What is happening as $n\to\infty$?

10. *Fast Fourier transform for DFT*. Consider DFT_N the discrete Fourier transform mod N with $N=2^m$ being a power of 2.
(a) For $\underline{a}=(a_0,\ldots,a_{N-1})$ show that direct computation of $DFT_N(\underline{a})$ by matrix multiplication requires $2N^2-N$ basic multiplication and addition operations between the matrix elements of DFT_N and the components of $\underline{a}$.
(b) Show that $DFT_N(\underline{a})$ can be expressed in terms of two applications of $DFT_{N/2}$. (Hint: consider separately the even and odd numbered components of $\underline{a}$). Using this decomposition show that DFT_N may be computed with $T(N)$ basic additions and multiplications where $T(N)$ has leading term $N\log_2 N$ i.e. exponentially faster as a function of m than the direct method of (a). Find the exact formula for $T(N)$.

11. *Parseval's relation*. By considering the the Fourier transform of the function $f(x)=\cos(x)$ for $|x|<\pi/2$ and $f(x)=0$ for $|x|\geq\pi/2$, and the Fourier transform of its derivative, show that

$$\int_0^\infty \frac{\frac{\pi^2}{4}\cos^2 t}{\left(\frac{\pi^2}{4}-t^2\right)^2}dt = \int_0^\infty \frac{t^2\cos^2 t}{\left(\frac{\pi^2}{4}-t^2\right)^2}dt = \frac{\pi}{4}.$$

12. *Laplace's equation*. Show that the inverse Fourier transform of the function (for any real α)

$$\tilde{f}_\alpha(k) = \begin{cases} e^{k\alpha}-e^{-k\alpha}, & |k|\leq 1,\\ 0 & |k|>1,\end{cases}$$

is

$$f_\alpha(x) = \frac{2i}{\pi(\alpha^2+x^2)}(\alpha\cosh\alpha\sin x - x\cos x\sinh\alpha).$$

Determine, by using Fourier transforms, the solution of Laplace's equation in the infinite strip $0\leq y\leq 1$, i.e.

$$\nabla^2\psi=0;\quad -\infty<x<\infty,\ 0<y<1,$$

where $\psi(x,0)=f_1(x)$ the function given above with $\alpha=1$, and $\psi(x,1)=0$ for $-\infty<x<\infty$.