Course C10—No. B8

METHODS — EXAMPLES III

Green's functions

1. *Initial value problem*. The reading $\theta(t)$ of an ammeter satisfies

$$\ddot{\theta} + 2p\dot{\theta} + (p^2 + q^2)\theta = f(t)\,,$$

where p, q are constants with $p > 0$. The ammeter is set so that θ and $\dot{\theta}$ are zero when $t = 0$. Assuming $q \neq 0$, show by constructing the Green's function that

$$\theta(t) = \frac{1}{q}\int_0^t e^{-p(t-\tau)} \sin[q(t-\tau)]f(\tau)d\tau\,.$$

Derive the same result using Fourier transforms, showing that the transfer function for this system is

$$\tilde{R}(\omega) = \frac{1}{2qi}\left[\frac{1}{(i\omega + p - qi)} - \frac{1}{(i\omega + p + qi)}\right].$$

2. *Boundary value problem*. Obtain the Green's function $G(x, \xi)$ satisfying

$$\frac{d^2G}{dx^2} - \lambda^2 G = \delta(x - \xi)\,, \qquad 0 \leq x \leq 1\,, \quad 0 \leq \xi \leq 1\,.$$

where λ is real, subject to the boundary conditions $G(0, \xi) = G(1, \xi) = 0$. Show that the solution to the equation

$$\frac{d^2y}{dx^2} - \lambda^2 y = f(x), \qquad \text{subject to the same boundary conditions is}$$

$$y = -\frac{1}{\lambda\sinh\lambda}\left\{\sinh\lambda x\int_x^1 f(\xi)\sinh\lambda(1-\xi)d\xi + \sinh\lambda(1-x)\int_0^x f(\xi)\sinh\lambda\xi d\xi\right\}.$$

3. *Finite asymptotics*. A linear differential operator is defined by

$$L_x y = -\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{dy}{dx}\right) + y\,.$$

By writing $y = z/x$ or otherwise, find those solutions of $L_x y = 0$ which are either (a) bounded as $x \to 0$, or (b) bounded as $x \to \infty$. Find the Green's function $G(x, a)$ satisfying

$$L_x G(x, a) = \delta(x - a)\,,$$

and both conditions (a) and (b). Use $G(x, a)$ to solve (subject to conditions (a) and (b))

$$L_x y(x) = \begin{cases} 1, & \text{for } 0 \leq x \leq R\,, \\ 0, & \text{for } x > R\,. \end{cases}$$

Show that the solution has the form, for suitable constants A, B

$$y(x) = \begin{cases} 1 + Ax^{-1}\sinh x\,, & \text{for } 0 \leq x \leq R\,, \\ Bx^{-1}e^{-x}\,, & \text{for } x > R\,. \end{cases}$$

4. *Higher order initial value problem**. Show that the Green's function for the initial value problem ($' \equiv \frac{d}{dt}$)

$$y'''' + k^2 y'' = f(t)\,, \qquad y(0) = y'(0) = y''(0) = y'''(0) = 0\,,$$

$$\text{is given by} \qquad G(t, \tau) = \begin{cases} 0\,, & 0 < t < \tau\,, \\ k^{-2}(t - \tau) - k^{-3}\sin k(t - \tau)\,, & t > \tau\,. \end{cases}$$

Hence write down the solution for $f(t) = e^{-t}$ and verify that it satisfies the equation and the initial conditions. [*Hint:* Make life easy by noting $G(\tau, \tau) = 0$ for an IVP Green's function and so use the time invariance of the equation to take $G(t, \tau) = f(t - \tau)$ for $t > \tau$.]

The Dirac delta function

5. *Delta function properties*. The function $\phi(x)$ is monotone increasing in $[a, b]$ and has a (simple) zero at $x = c$ (i.e. $\phi'(c) \neq 0$) where $a < c < b$. Show that

$$\int_a^b f(x)\delta[\phi(x)]dx = \frac{f(c)}{|\phi'(c)|} \ .$$

Show that the same formula applies if $\phi(x)$ is monotone decreasing and hence derive a formula for general $\phi(x)$ provided the zeros are simple. Deduce that $\delta(at) = \delta(t)/|a|$ for $a \neq 0$. Also establish that

$$\int_{-\infty}^{+\infty} |x|\delta(x^2 - a^2)dx = 1 \quad .$$

6. *Delta function derivative**. Show using polar coordinates that

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x^2+y^2)\delta'(x^2+y^2-1)\delta(x^2-y^2)dx\,dy = f(1) - f'(1)\,.$$

Fourier transforms

7. *Fourier transforms of functions of finite extent*. Calculate the Fourier transforms of the following functions. All are non-zero only on the interval $|x| < c$, and zero elsewhere.

$f(x) = 1$.
$f(x) = e^{iax}$.
$f(x) = \sin(ax)$.
$f(x) = \cos(ax)$.

8. *Functions with discontinuities*. Let $f(x) = e^{-x}$ for $0 < x < \infty$, and $f(x) = 0$ for $x < 0$. Show that

$$\tilde{f}(k) = \frac{1 - ik}{1 + k^2}.$$

Show that the inverse Fourier transform of this Fourier transform $\tilde{f}(k)$ takes the value of $1/2$ at $x = 0$. (This is a general property of Fourier transforms, analogously to Fourier series. Inversion for general x is really straightforward with Complex Methods.)

9. *Fourier transform of Gaussians*. By using differentiation and the shift property, calculate the Fourier transform of a Gaussian distribution with a peak at $\mu \neq 0$, i.e. $f(x) = \exp[-n^2(x-\mu)^2]$.
Now let $\mu = 0$, and consider $\delta_n(x) = (n/\sqrt{\pi})f(x)$. Sketch $\delta_n(x)$ and $\tilde{\delta}_n(k)$ for small and large n. What is $\int_{-\infty}^{\infty} \delta_n(x)dx$? What is happening as $n \to \infty$?

10. *Parseval's relation for the discrete Fourier transform*. Using the notation of the lecture notes, prove Parseval's relation for the DFT:

$$\sum_{m=0}^{N-1} |h(t_m)|^2 = \frac{1}{N}\sum_{n=0}^{N-1} |\tilde{h}_d(f_n)|^2.$$

11. *Parseval's relation continued*. By considering the the Fourier transform of the function $f(x) = \cos(x)$ for $|x| < \pi/2$ and $f(x) = 0$ for $|x| \geq \pi/2$, and the Fourier transform of its derivative, show that

$$\int_0^\infty \frac{\frac{\pi^2}{4}\cos^2 t}{\left(\frac{\pi^2}{4} - t^2\right)^2}dt = \int_0^\infty \frac{t^2\cos^2 t}{\left(\frac{\pi^2}{4} - t^2\right)^2}dt = \frac{\pi}{4}.$$

12. *Laplace's equation*. Show that the inverse Fourier transform of the function

$$\tilde{f}(k) = \begin{cases} e^k - e^{-k}, & |k| \leq 1, \\ 0 & |k| > 1, \end{cases}$$

is

$$f(x) = \frac{2i}{\pi(1+x^2)}(\cosh 1 \sin x - x\cos x \sinh 1).$$

Determine, by using Fourier transforms, the solution of Laplace's equation in the infinite strip $0 \leq y \leq 1$, i.e.

$$\nabla^2\psi = 0; \quad -\infty < x < \infty, \ 0 < y < 1,$$

where $\psi(x, 0) = f(x)$ the function given above, and $\psi(x, 1) = 0$ for $-\infty < x < \infty$.
(This was a long tripos question (2004/4/II/15A) for Complex Methods on material now in the Methods schedule.)

[†] If you find any errors in the Methods Examples sheets, please inform your supervisor or email c.p.caulfield@bpi.cam.ac.uk.