Mathematical Tripos Part IB — **Michaelmas Term 2017**
Quantum Mechanics — **Dr. J.M. Evans**

Example Sheet 1

Values of some physical constants are given in the supplementary notes

1. When a sample of potassium is illuminated with light of wavelength 3×10^{-7}m, electrons are emitted with kinetic energy 2.1 eV. When the same sample is illuminated with light of wavelength 5×10^{-7}m it emits electrons with kinetic energy 0.5 eV. Use Einstein's explanation of this *photoelectric effect* to obtain a value for Planck's constant, and find the minimum energy W needed to free an electron from the surface of potassium.

2. The light from a faint star has energy flux $10^{-10}\,\mathrm{Jm^{-2}s^{-1}}$. If the wavelength of the light is 5×10^{-7}m, estimate the number of photons from this star that enter a human eye in one second.

3. Consider the Bohr model of the Hydrogen atom, taking the electron to be a non-relativistic point particle of mass m travelling with speed v in a circular orbit of radius r around a point-like proton. The inward acceleration v^2/r must be provided by the Coulomb attraction $e^2/4\pi\epsilon_0 r^2$, and the angular momentum is assumed to be quantised: $L = mvr = n\hbar$, with $n = 1, 2, 3, \ldots$. Show that the possible energy levels for the electron are

$$E = -\frac{1}{2}mc^2\alpha^2\frac{1}{n^2}$$

where α is the fine structure constant.

(i) Is the result for v consistent with the assumption that the motion of the electron is non-relativistic?

(ii) Suppose that the electron makes a transition from one energy level to another, emitting a photon in the process. What is the smallest possible wavelength for the emitted photon, and how does this compare to the Bohr radius r_1 (corresponding to $n = 1$)?

4. A muon is a particle with the same charge as an electron, but with a mass about 207 times larger. A muon can be captured by a proton to form an atom of muonic Hydrogen. How does the radius of the $n = 1$ state orbit compare to that of ordinary Hydrogen?

5. The time-independent Schrödinger Equation for a one-dimensional harmonic oscillator is

$$-\frac{\hbar^2}{2m}\frac{\mathrm{d}^2\psi}{\mathrm{d}x^2} + \frac{1}{2}Kx^2\psi \;=\; E\psi$$

where m is the mass and the constant K gives the strength of the restoring force. Verify that there are energy eigenfunctions of the form

$$\psi_0(x) \;=\; C_0\, e^{-x^2/2\alpha}\;, \qquad \psi_1(x) \;=\; C_1\, xe^{-x^2/2\alpha}$$

for a certain value of α, to be determined, and find the corresponding energy eigenvalues E_0 and E_1. Sketch ψ_0, ψ_1, and $V(x)$. (C_0 and C_1 are normalisation constants that you need not determine.)

6. For the harmonic oscillator in Example **5**, write down the time-dependent Schrödinger Equation satisfied by $\Psi(x,t)$ and give the solution for each of the following initial conditions:

$$\text{(i)}\quad \Psi(x,0) = \psi_0(x)\,, \qquad \text{(ii)}\quad \Psi(x,0) = \psi_1(x)\,, \qquad \text{(iii)}\quad \Psi(x,0) = \tfrac{1}{2}(\,\sqrt{3}\,\psi_0(x) - i\,\psi_1(x)\,)\,.$$

For the solution in case (iii), what is the first time $T > 0$ at which $\Psi(x,T)$ and $\Psi(x,0)$ correspond to physically equivalent states?

7. A particle of mass m moves freely in one dimension ($V = 0$). Consider the wavefunction

$$\Psi(x,t) \;=\; C\,\gamma(t)^{-1/2}\,\exp(\,-x^2/2\gamma(t)\,)\,,$$

where $\gamma(t)$ is complex-valued and C is a constant. By substituting into the time-dependent Schrödinger equation, find a necessary and sufficient condition on $\gamma(t)$ for $\Psi(x,t)$ to be a solution. Hence determine $\gamma(t)$ if $\gamma(0) = \alpha$, a real positive constant.

Write down and simplify an expression for the probability density for the particle at time t and find a value for the constant C such that $\Psi(x,t)$ is normalised. Comment briefly on the behaviour of the probability density as t increases.

8. Consider a particle in one dimension in a potential $V(x)$ that tends to zero rapidly as $x \to \pm\infty$.

(i) Let $\psi_1(x)$ and $\psi_2(x)$ be normalisable energy eigenfunctions of the Hamiltonian with the same energy eigenvalue E. By considering $\psi_1\,\psi'_2 - \psi_2\,\psi'_1$, show that ψ_1 and ψ_2 must be proportional to one another. What does this mean, physically?

(ii) Can the wavefunction for a normalised energy eigenstate always be chosen to be real?

(iii) Show that if $\psi(x)$ is any normalised energy eigenstate then $\langle\,\hat{p}\,\rangle_\psi = 0\,$.

9. Write down the time-independent Schrödinger equation for the wavefunction of a particle moving in a potential $V = -U\delta(x)$, where U is a positive constant and $\delta(x)$ is the Dirac delta function. Integrate the equation over the interval $-\epsilon < x < \epsilon$, for a positive constant ϵ, and hence deduce that there is a discontinuity at $x = 0$ in the derivative of $\psi(x)$:

$$\lim_{\epsilon\to 0}\,[\,\psi'(\epsilon) - \psi'(-\epsilon)\,] \;=\; -\frac{2mU}{\hbar^2}\psi(0)\ .$$

By using this condition to relate appropriate solutions for $x > 0$ and $x < 0$, find the unique normalisable eigenstate of the Hamiltonian, and determine its energy eigenvalue.

10. Consider a square well potential with $V(x) = -U$ for $|x| < a$ and $V(x) = 0$ otherwise (U is a positive constant). Show that there are no bound states (normalisable energy eigenfunctions) which satisfy $\psi(-x) = -\psi(x)$ (i.e. which have odd parity) if $a^2U < (\pi\hbar)^2/8m$.

11. Sketch the potential

$$V(x) = -\frac{\hbar^2}{m}\mathrm{sech}^2 x\ .$$

Show that the time-independent Schrödinger equation for a particle in this potential can be written

$$A^\dagger A\,\psi \;=\; (\mathcal{E}+1)\psi$$

where $\mathcal{E} = 2mE/\hbar^2$ and

$$A = \frac{\mathrm{d}}{\mathrm{d}x} + \tanh x\ , \quad A^\dagger = -\frac{\mathrm{d}}{\mathrm{d}x} + \tanh x\ .$$

Show, by integrating by parts, that for any normalised wavefunction ψ,

$$\int_{-\infty}^{\infty} \psi^* A^\dagger A\psi\,\mathrm{d}x \;=\; \int_{-\infty}^{\infty} (A\psi)^*(A\psi)\,\mathrm{d}x$$

and deduce that the eigenvalues of $A^\dagger A$ are non-negative. Hence show that the ground state (with lowest energy) has $\mathcal{E} \geq -1$. Show that a wavefunction $\psi_0(x)$ is an energy eigenstate with $\mathcal{E} = -1$ iff

$$\frac{\mathrm{d}\psi_0}{\mathrm{d}x} + \tanh x\,\psi_0 \;=\; 0\ .$$

Find and sketch $\psi_0(x)$.

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