Example Sheet 2

1. A particle of mass m is confined to a one-dimensional box $0 \leq x \leq a$ (the potential $V(x)$ is zero inside the box, and infinite outside). Show that the energy eigenvalues are $E_n = \hbar^2\pi^2 n^2/2ma^2$ for $n = 1, 2, \ldots,$ and determine corresponding normalised energy eigenstates $\psi_n(x)$. Show that the expectation value and the uncertainty for a measurement of $\hat{x}$ in the state ψ_n are

$$\langle \hat{x} \rangle_n = \frac{a}{2} \qquad \text{and} \qquad (\Delta x)_n^2 = \frac{a^2}{12}\Big(1 - \frac{6}{\pi^2 n^2}\Big) \, .$$

Does the limit $n \to \infty$ agree with what you would expect for a classical particle in this potential?

2. Write down the Hamiltonian H for a harmonic oscillator of mass m and frequency ω. Express $\langle H \rangle$ in terms of $\langle \hat{x} \rangle$, $\langle \hat{p} \rangle$, Δx and Δp, all defined for some normalised state ψ. Use the Uncertainty Relation to deduce that $E \geq \frac{1}{2}\hbar\omega$ for any energy eigenvalue E.

3. Let $\Psi(x,t)$ be a solution of the time-dependent Schrödinger Equation with zero potential (corresponding to a free particle). Show that

$$\Phi(x,t) = \Psi(x - ut, t)\, e^{ikx} e^{-i\omega t}$$

is also a solution if the real constants u, k and ω satisfy certain conditions, to be specified. Express $\langle \hat{x} \rangle_\Phi$ and $\langle \hat{p} \rangle_\Phi$ in terms of $\langle \hat{x} \rangle_\Psi$ and $\langle \hat{p} \rangle_\Psi$ at each fixed time t.

4. The energy levels of the harmonic oscillator are $E_n = (n + \frac{1}{2})\hbar\omega$ for $n = 0, 1, 2, \ldots$ and the corresponding stationary state wavefunctions are

$$\psi_n(x) \;=\; h_n(y) e^{-y^2/2} \qquad \text{where} \qquad y = (m\omega/\hbar)^{1/2} x$$

and h_n is a polynomial of degree n with $h_n(-y) = (-1)^n h_n(y)$. Use the orthogonality relation

$$(\psi_m, \psi_n) = \delta_{mn}$$

to determine ψ_2 and ψ_3 up to overall constants.

Give an expression for the quantum state of the oscillator $\Psi(x,t)$ if the initial state is $\Psi(x,0) = \sum_{n=0}^{\infty} c_n \psi_n(x)$, where c_n are complex constants. Deduce that

$$|\, \Psi(\, x,\, 2p\pi/\omega\,)\,|^2 = |\, \Psi(\, -x,\, (2q{+}1)\pi/\omega\,)\,|^2$$

for any integers p, $q \geq 0$. Comment on this result, considering the particular case in which $\Psi(x,0)$ is sharply peaked around position $x = a$.

5. Consider the Schrödinger Equation in one dimension with potential $V(x)$. Show that for a stationary state, the probability current J is independent of x.

Now suppose that an energy eigenstate $\psi(x)$ corresponds to scattering by the potential and that $V(x) \to 0$ as $x \to \pm\infty$. Given the asymptotic behaviour

$$\psi(x) \,\sim\, e^{ikx} + Be^{-ikx} \quad (x \to -\infty) \qquad \text{and} \qquad \psi(x) \,\sim\, Ce^{ikx} \quad (x \to +\infty)$$

show that $|B|^2 + |C|^2 = 1$. How should this be interpreted?

6. A particle is incident on a potential barrier of width a and height U. Assuming that $U = 2E$, where $E = \hbar^2 k^2/2m$ is the kinetic energy of the incident particle, find the transmission probability.
[*Work through the algebra, which simplifies in this case, rather than quoting the general result.*]

7. Consider the time-independent Schrödinger Equation with potential $V(x) = -U\delta(x)$; the wavefunction ψ is continuous, but satisfies $\psi'(0+) - \psi'(0-) = -(2mU/\hbar^2)\,\psi(0)$. [*Recall Example* **9** *on Sheet* 1.] Show that there is a scattering solution with energy eigenvalue $E = \hbar^2k^2/2m$ for any real $k > 0$ and find the transmission and reflection amplitudes $A_{\rm tr}(k)$ and $A_{\rm ref}(k)$.

If k is allowed to take complex values in the solution above, is it still an eigenfunction of the Hamiltonian? Show that $A_{\rm tr}(k)$ and $A_{\rm ref}(k)$ are singular at $k = i\kappa$ for a certain real, positive value of κ. Use this observation to find a bound state (normalisable) solution in the potential by first re-scaling the scattering solution and then setting $k = i\kappa$. What is the energy of this bound state?

8. A particle of mass m is in a one-dimensional infinite square well (a potential box) with $V = 0$ for $0 < x < a$ and $V = \infty$ otherwise. The normalised wavefunction for the particle at time $t = 0$ is

$$\Psi(x, 0) = Cx(a - x)\ .$$

(i) Determine the real constant C.

(ii) By expanding $\Psi(x, 0)$ as a linear combination of energy eigenfunctions (found in Example **1** above), obtain an expression for $\Psi(x, t)$, the wavefunction at time t.

(iii) A measurement of the energy is made at time $t > 0$. Show that the probability that this yields the result $E_n = \hbar^2\pi^2n^2/2ma^2$ is $960/\pi^6n^6$ if n is odd, and zero if n is even. Why should the result for n even be expected? Which value of the energy is most likely, and why is its probability so close to unity?

9. A quantum system has Hamiltonian H with normalised eigenstates ψ_n and corresponding energies E_n $(n = 1, 2, 3, \ldots)$. A linear operator Q is defined by its action on these states:

$$Q\psi_1 = \psi_2\ , \qquad Q\psi_2 = \psi_1\ , \qquad Q\psi_n = 0 \quad n > 2\ .$$

Show that Q has eigenvalues ± 1 (in addition to zero) and find the corresponding normalised eigenstates $\chi_\pm$, in terms of energy eigenstates. Calculate $\langle H\rangle$ in each of the states $\chi_\pm$.

A measurement of Q is made at time zero, and the result $+1$ is obtained. The system is then left undisturbed for a time t, at which instant another measurement of Q is made. What is the probability that the result will again be $+1$? Show that the probability is zero if the measurement is made when a time $T = \pi\hbar/(E_2 - E_1)$ has elapsed (assume $E_2 - E_1 > 0$).

10. In the previous example, suppose that an experimenter makes n successive measurements of Q at regular time intervals T/n. If the result $+1$ is obtained for one measurement, show that the probability that the next measurement also gives $+1$ can be written as $|A_n|^2$, with

$$A_n = 1 - \frac{iT(E_1 + E_2)}{2\hbar n} + \mathcal{O}\Big(\frac{1}{n^2}\Big)$$

The probability that all n measurements give the result $+1$ is therefore $P_n = (|A_n|^2)^n$. Show that

$$\lim_{n\to\infty} P_n = 1\ .$$

Interpreting $\chi_\pm$ as the 'not-boiling' and 'boiling' states of a two-state 'quantum kettle', this shows that a watched quantum kettle never boils (also called the Quantum Zeno Paradox).

11. Let H be a Hamiltonian and ψ any normalised eigenstate with energy E. Show that, for any operator A,

$$\langle\, [H, A]\,\rangle_\psi \;=\; 0\,.$$

For a particle in one dimension, let $H = T + V$ where $T = \hat{p}^2/2m$ is the kinetic energy and $V(\hat{x})$ is any (real) potential. By setting $A = \hat{x}$ in the result above and using the canonical commutation relation between position and momentum, show that $\langle\hat{p}\rangle_\psi = 0$.

Now assume further that $V(\hat{x}) = k\hat{x}^n$ (with k and n constants). By taking $A = \hat{x}\hat{p}$, show that

$$\langle T\rangle_\psi = \frac{n}{n+2}\,E \qquad \text{and} \qquad \langle V\rangle_\psi = \frac{2}{n+2}\,E\,.$$