Mathematical Tripos Part 1B Dr. Eugene A. Lim
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QUANTUM MECHANICS

Example Sheet 2

1. Show that the probability current j for a stationary state $\psi(x)$ of a particle scattering off an arbitrary potential $V(x)$ in one dimension is independent of x. Given that $\psi(x)$ has the asymptotic behaviour

$$\psi(x) = \begin{cases} e^{ikx} + Ae^{-ikx} & x \ll 0 \\ Be^{ikx} & x \gg 0 \end{cases},$$

show that $|A|^2 + |B|^2 = 1$. How should you interpret this?

2. A particle is incident on a square potential barrier of width a and height U_0. Assuming that $U_0 = 2E$, where $E = \hbar^2 k^2/2m$ is the kinetic energy of the incident particle, find the transmission probability. [*You should work from first principles rather than quote formulae from the lectures because the algebra simplifies in this special case.*]

3. Consider the time-independent Schrödinger equation in Q.8, Ex.Sheet 1. Show that for any real k,

$$\psi(x) = e^{ikx}(\tanh x - ik)$$

is a solution, and find its (scaled) energy ε. Show that this is the wavefunction of a scattering state where the reflection probability vanishes. Find the transmission amplitude, and verify that the transmission probability is 1.

*Consider the extension (analytic continuation) of this solution to positive imaginary values of k. Show that for the value of k corresponding to the bound state at $\varepsilon = -1$, $\psi(x)$ reduces to the bound state wavefunction, and that the transmission amplitude has a simple pole.

4. Show that the energy levels of the one-dimensional harmonic oscillator of angular frequency ω are $E_n = \left(n + \frac{1}{2}\right)\hbar\omega$. Find the wavefunctions $\psi_0(x)$,$\psi_1(x)$ and $\psi_2(x)$ corresponding, respectively, to $n = 0, 1$ and 2. Verify that ψ_0 and ψ_2 have even parity and that ψ_1 has odd parity. Use this to deduce that ψ_1 is orthogonal to both ψ_0 and ψ_2. Verify that ψ_0 and ψ_2 are also orthogonal, i.e.

$$\int_{-\infty}^{\infty} \psi_2^* \psi_0 \, dx = 0\,.$$

5. What condition must an operator A satisfy to be Hermitian? Show that the expectation value of a Hermitian operator is real. Show that $i[A_1, A_2]$ is Hermitian if A_1 and A_2 are.

6. A particle of mass m is in a one-dimensional infinite square well, with $U = 0$ for $0 < x < a$ and $U = \infty$ otherwise. Show that its energy eigenstates have energies $E_n = (\hbar\pi n)^2/2ma^2$ for positive integer n.

The normalized wavefunction of the particle at time $t = 0$ is

$$\psi(x, 0) = Cx(a - x)\,.$$

Determine the real constant C. Determine $\psi(x, t)$, the wavefunction at time t. [*Write* $\psi(x, 0)$ *as a linear combination of normalized energy eigenstates; i.e., as a Fourier series.*] A measurement of the energy E is made at time $t > 0$. Show that the probability that this yields E_n for even n is zero. Why is this? Show that the probability that the measurement yields E_n is $960/\pi^6 n^6$ for odd n. Which value of E is the most likely and why is its probability so close to unity?

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7. A particle is confined to the one-dimensional box $0 < x < a$. If you have not already done this for Q.6, show that the energy levels are proportional to n^2 for positive integer n, and find the corresponding complete set of normalized stationary states $\psi_n(x)$. Let $\langle A\rangle_n$ denote the expectation value of any operator A in the state ψ_n. Show that

$$\langle x\rangle_n = \frac{1}{2}a\,, \qquad \langle (x-\langle x\rangle_n)^2\rangle_n = \frac{a^2}{12}\left(1-\frac{6}{n^2\pi^2}\right).$$

Hence show that the classical expectation values, i.e. with the particle bouncing back and forth and equally likely to be anywhere in the box, are recovered in the $n\to\infty$ limit.

8. A 'two-state' quantum system has orthonormal energy eigenstates ψ_1 and ψ_2, with energy eigenvalues E_1 and $E_2 = E_1 + \Delta E$ ($\Delta E > 0$). These energy eigenstates form a complete set of wavefunctions for the system. Let S be a linear operator such that $S\psi_1 = \psi_2$ and $S\psi_2 = \psi_1$. Show that the eigenvalues of S are ± 1 and write down the corresponding normalized eigenfunctions $\phi_\pm$ in terms of the energy eigenstates. Compute the expectation values $\langle E\rangle_\pm$ of the energy in the states $\phi_\pm$.

The observable corresponding to S is measured and the value $+1$ is found. The system is then left undisturbed for a time t, after which S is measured again. What is the probability that the measured value of S will again be $+1$. Show that this probability vanishes when $t = T \equiv \pi\hbar/\Delta E$.

*In a second run of this experiment it is decided to measure S at a large number n of small time intervals T/n. Each measurement yields either $+1$ or -1, with the wavefunction being reset at ϕ_+ or ϕ_-, respectively, by the measurement. Show that the probability amplitude for the state to be found in the $+1$ eigenstate after a time interval T/n, given that it started in this eigenstate, is

$$A_n = 1 - \frac{i}{n\hbar}\,T\langle E\rangle_+ + \mathcal{O}(\frac{1}{n^2})\,.$$

The probability that all n measurements of S will yield the value $+1$ is therefore $P_n = (|A_n|^2)^n$. Show that

$$\lim_{n\to\infty} P_n = 1\,.$$

[If you interpret ϕ_+ and ϕ_- to be the 'not boiling' and 'boiling' states of a two-state 'quantum kettle' then you have just proved that *a watched kettle never boils*. This is also known as the *quantum Zeno effect*. Note that it is a real physical effect, in contrast to the ancient (so-called) Zeno paradox.]

9. The Hamiltonian operator for a particle in one dimension is $H = T + U$ where $T = p^2/2m$, and U is any potential. Show that the expectation value $\langle T\rangle$ is positive in any (normalized) state. By considering $\langle H\rangle$, show that the energy of the lowest bound state (assuming there is one) has energy above the minimum of U.

Suppose ψ is an eigenstate of H with energy E. Show that, for any operator A, and in the state ψ

$$\langle [H, A]\rangle = 0\,.$$

By taking $A = x$, show that $\langle p\rangle = 0$. Now let $U(x) = kx^n$ for constants k and n; by taking $A = xp$ derive the *virial theorem*

$$2\langle T\rangle = n\langle U\rangle\,.$$

Hence show that

$$\langle T\rangle = \frac{n}{n+2}E\,.$$

10. A particle of mass m moves in one dimension subject to the potential $U(x) = \frac{1}{2}m\omega^2x^2$. Express the expectation value of the energy E in terms of $\langle x\rangle$, $\langle p\rangle$, Δx and Δp. Hence show, using the uncertainty relation for x and p, that in any state

$$\langle E\rangle \geq \frac{1}{2}\hbar\omega\,.$$

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11. (Optional) *The Beam Interpretation.* Instead of interpreting the plane-wave solution, with $k^2 \equiv |\mathbf{k}|^2$

$$\psi(\mathbf{x}, t) = A \exp[i\mathbf{k} \cdot \mathbf{x}] \exp\left[-\frac{i\hbar k^2 t}{2m}\right]$$

as describing the wavefunction of a particle, we can interpret it as describing a beam of particles with momentum $\mathbf{p} = \hbar\mathbf{k}$ and energy $E = |\mathbf{p}|^{\mathbf{2}}/2m$ instead. In this interpretation, we think of the probability density function $\rho(\mathbf{x}, \mathbf{t})$ as the *average* density of particles. Show that the density current $\mathbf{j}$ is then given by

$$\mathbf{j}(\mathbf{x}, t) = \textit{average density} \times \textit{velocity}$$

i.e. the number of particles that flow through a unit area per time (flux).

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