EXAMPLES II

1. Yukawa potential and Seelinger's paradox: Suppose that we have a point at radius $\mathbf{r}_0$ (with $r_0 = |\mathbf{r}_0|$) located inside a sphere of constant density ρ and radius $R > r_0$ centred on the origin O. The Newtonian potential at this point is $\Phi(r_0) = 2\pi G\rho(R^2 - r_0^2/3)$ which grows without limit as the sphere's radius increases, $R \to \infty$. In the context of an infinite Euclidean universe, Seelinger regarded this as creating "insurmountable difficulties", so he proposed that gravity had a finite range. His new gravitational potential for a point mass M satisfied a modified Poisson equation $\nabla^2\Phi - \lambda^2\Phi = 4\pi\rho$, taking the form:

$$\Phi(r) = GM\frac{e^{-\lambda r}}{r}\,.$$

Show that the external potential of a thin uniform spherical shell of radius $r = a$ is the same as that of a point mass located at its centre at $r = 0$ but with a mass equal to $\sinh(\lambda a)/\lambda a$ times the mass of the shell. What happens as $\lambda \to 0$?

2. Accelerating universe For a flat universe ($k = 0$), solve the Friedmann equation for matter with an equation of state $P = w\rho c^2$ to find

$$a(t) = \left(\frac{t}{t_0}\right)^{2/3(w+1)}.$$

The 'deceleration parameter' q_0 is defined as $q(t_0)$, where $q(t) = -\ddot{a}\,a/\dot{a}^2$. Show, in this case, that $q = \frac{1}{2}(3w+1)$. For a universe (arbitrary k) with several matter components $\Omega_{\rm M}$, $\Omega_{\rm R}$, Ω_Λ (with $w = 0$, $1/3$, -1 respectively), show that the deceleration parameter today is

$$q_0 = \tfrac{1}{2}\Omega_{\rm M} + \Omega_{\rm R} - \Omega_\Lambda\,.$$

Expand the scale factor $a(t)$ about $t = t_0$, to find that

$$\frac{1}{1+z} \;\equiv\; \frac{a(t)}{a(t_0)} \;=\; 1 + H_0(t-t_0) - \tfrac{1}{2}q_0H_0^2(t-t_0)^2 \;+\ldots$$

Hence, by considering the physical distance a photon travels in a given time t, substitute $t \to r$ to find the measurable relation:

$$cz \;=\; H_0 r + \tfrac{1}{2}(1+q_0)H_0^2 r^2 + \ldots\,.$$

3. Dark energy: Our universe today is believed to be flat ($k = 0$) and filled with two major components:

pressure-free matter ($P_{\rm M} = 0$) and dark energy with equation of state $P_{\rm Q} = -\rho_{\rm Q}c^2$ with density parameters today given respectively by $\Omega_{\rm M} = \rho_{\rm M}(t_0)/\rho_c(t_0)$ and $\Omega_\Lambda = \rho_{\rm Q}(t_0)/\rho_c(t_0)$. Assume that each component independently satisfies the fluid conservation equation to show that the total mass density can be expressed as

$$\rho(t) = \left(\Omega_{\rm M}\left(\frac{a_0}{a}\right)^3 + \Omega_\Lambda\right)\frac{3H_0^2}{8\pi G}\,,$$

where H_0 is the Hubble parameter measured today.

Now consider the substitution $b = a^{3/2}$ in the Friedmann equation to show that the solution for the scale factor can be written in the form

$$a(t) = \beta(\sinh\alpha t)^{2/3}\,,$$

where α and β are constants which you should specify in terms of $\Omega_{\rm M}$, Ω_Λ and H_0. [*Hint:* Recall that $\int dx/\sqrt{x^2+1} = \sinh^{-1}x\,.$]

Estimate the time when deceleration turns into acceleration $\ddot{a} = 0$ (i.e. Λ dominates), taking $\Omega_{\rm M} = 0.3$, $\Omega_\Lambda = 0.7$ and $H_0^{-1} = 14$ Gyrs. Find the age of the universe t_0. Verify the expected asymptotic behaviour for ($t \to 0$ (EdS) and $t \to \infty$ (de Sitter).

4 . Equation of state of the vacuum: The energy-momentum tensor for a fluid with pressure P and density ρ in its rest frame is

$$T_{ab} = \begin{bmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

Show that after applying a Lorentz boost in the x-direction at velocity $v = \beta c$ we get

$$T'_{ab} = \begin{bmatrix} \gamma^2\rho c^2 + \gamma^2\beta^2 P & \gamma^2\beta(\rho c^2 + P) & 0 & 0 \\ \gamma^2\beta(\rho c^2 + P) & \gamma^2\beta^2\rho c^2 + \gamma^2 P & 0 & 0 \\ 0 & 0 & P & 0 \\ 0 & 0 & 0 & P \end{bmatrix}$$

Show that the requirement that the components of the energy-momentum tensor of the vacuum be the same in different inertial frames requires the equation of state to be $P = -\rho c^2$. Why is this invariance property desirable for a vacuum energy density? [Note that this 'fluid' is equivalent to a cosmological constant, Λ, in the cosmological equations.]

5. Inflation and the flatness problem Suppose that the universe "reheats" instantaneously after a period of inflation ($a \propto \exp(Ht)$ with H constant), restarting the standard Hot Big Bang with an effective initial time $t_{\rm reh} \approx 10^{-35}$s (assume this universe is filled only with matter and radiation with $1 + z_{\rm eq} \approx 10^5$, $\Omega_0 \approx 1.02$ and $t_0 \approx 10^{18}$s). Show that at a time $t_{\rm reh}$, the density parameter must be fine-tuned to approximately $\Omega_{\rm reh} - 1 \approx 10^{-52}$. How much expansion during inflation is required to solve this flatness problem, that is, estimate the number N of e-folds and the time interval for inflation $\Delta t = t_{\rm reh} - t_{\rm i}$? Plot $\log\Omega$ vs $\log(t - t_i)$ to illustrate how the flatness problem is cured.

6. A simple inflationary model in the slow-roll approximation: In the very early universe, where we take the curvature $k = 0$, suppose that we have a homogeneous scalar field ϕ (the inflaton) with a vacuum potential energy $\epsilon_{\rm vac} = \rho_{\rm vac} c^2 = V(\phi)$, with

$$V(\phi) = \frac{m^2c^2}{2\hbar^2}\phi^2 \equiv \tfrac{1}{2}\mathcal{M}^2\phi^2 \,.$$

(*Natural units* in which we set $\hbar = c = k_{\rm B} = 1$ are generally more convenient in this context.) The inflaton ϕ obeys the scalar wave equation (or Klein-Gordon equation),

$$\ddot{\phi} + 3H\dot{\phi} + c^2\frac{dV}{d\phi} = 0 \,. \qquad (*)$$

However, during inflation (after starting with a large initial $\phi_{\rm i}$) we have overdamped evolution satisfying the so-called "slow-roll" conditions, $|\ddot{\phi}| \ll |3H\dot{\phi}|$ and $\frac{1}{2}\dot{\phi}^2 \ll V(\phi)$. This means that eqn $(*)$ and the Friedmann equation become

$$3H\dot{\phi} \approx = -c^2\tfrac{dV}{d\phi} = -c^2\mathcal{M}^2\phi \,, \qquad H^2 \approx \tfrac{8\pi G}{3}\rho_{\rm vac} = \tfrac{\mathcal{M}^2c^2}{M_{\rm pl}^2}\phi^2 \,,$$

where $M_{\rm pl} = \sqrt{3\hbar c/4\pi G}$ is the Planck mass. Solve these equations to find the approximate slow-roll inflationary solution

$$\begin{aligned} \phi(t) &= \phi_i - \frac{1}{3}\mathcal{M}M_{\rm pl}ct \\ a(t) &= \exp\left[\frac{\mathcal{M}}{M_{\rm pl}}(\phi_i ct - \tfrac{1}{6}\mathcal{M}M_{\rm pl}c^2t^2)\right] = \exp\left[\frac{3}{2M_{\rm pl}^2}(\phi_i^2 - \phi(t)^2)\right] \,. \end{aligned}$$

Show that this solution only satisfies both "slow-roll" conditions while $|\phi| > \phi_{\rm reh}$, the value of which you should estimate (i.e. inflation only occurs if we choose $|\phi_{\rm i}| > \phi_{\rm reh}$ and it ends when $|\phi| \approx \phi_{\rm reh}$). How large must we take ϕ_i to solve the flatness problem if 60 e-folds of inflation are needed (Q5)?

7. An exact inflationary solution: A scalar field has an exponential potential of the form ($\hbar = c = 1$)

$$V(\phi) = V_0 e^{-\lambda\phi} \,,$$

where V_0 and λ are positive constants. Show that there is an exact solution of the $k = 0$ Friedmann universe when this scalar field is the only matter source with

$$\begin{aligned} a(t) &\propto t^{2/\lambda^2} \,, \\ \phi(t) &= \phi_0 + \frac{2}{\lambda}\ln(t) \,, \end{aligned}$$

where the constants λ, V_0 and ϕ_0 are related. Find that relation between them. What is the numerical condition on λ for inflation to occur? Evaluate the pressure-density ratio, p_ϕ/ρ_ϕ of the scalar field. What does your condition on λ for inflation to occur require for the value of this ratio?

8. Microstate counting and equilibrium distributions: N equal mass particles of total energy E populate a set of degenerate energy eigenstates with energies E_i and degeneracies g_i ($i = 1, 2, 3, \ldots, \infty$). The set $\{n\}$ of numbers n_i of particles with energy E_i is assigned a weight of the form

$$\Omega(\{n\}) = \prod_i W(n_i, g_i)\,. \qquad (*)$$

The most probable distribution $\{\bar{n}\}$ is obtained by maximising $\log \Omega$ subject to the constraints of fixed particle number N and fixed total energy E. Show that $\bar{n}_i$ is found by solving the equation

$$\frac{\partial \log W(n_i, g_i)}{\partial n_i} = \alpha + \beta E_i$$

where α and β are constants such that $\sum_i \bar{n}_i = N$ and $\sum_i \bar{n}_i E_i = E$. Write out this equation for each of the following three choices of the function W:

$$(i)\ W(n,g) = \frac{(g+n-1)!}{n!(g-1)!}\,, \qquad (ii)\ W(n,g) = \frac{g!}{n!(g-n)!}\,, \qquad (iii)\ W(n,g) = \frac{g^n}{n!}\,.$$

Assuming $g \gg 1$, $n \gg 1$, and $g \geq n$, use Stirling's formula [$\log n! = n \log n - n + \mathcal{O}(\log n)$] to simplify your result. Hence show that if α and β are appropriately related to the chemical potential μ and temperature T then $\{\bar{n}\}$ is the equilibrium distribution found in the lectures for a gas of (i) Bose-Einstein, (ii) Fermi-Dirac, and (iii) Maxwell-Boltzmann type. [The gas particles are said to obey BE, FD or MB 'statistics', respectively.]

9. Indistinguishable particles*: Assuming that $\Omega(\{n\})$ of the previous question equals the number of microstates available to the N particles for a given occupation number distribution $\{n\}$, explain why $\Omega(\{n\})$ must take the form $(*)$ if the N particles are identical.

Show that Ω *is* equal to the number of available microstates in cases (i) and (ii) assuming Bose-Einstein statistics and Fermi-Dirac statistics, respectively. [*Hint*: Consider how many different ways there are of painting n identical balls in g colours assuming (i) no restriction on the number of times each colour is used or (ii) that no colour may be used more than once.]

Show that in case (iii) Ω is $1/N!$ times the number of microstates available to N *distinguishable* particles. [This fact is related to the 'Gibbs paradox' of classical statistical mechanics.]

10. Relativistic pressure*: Evaluate the expression for the pressure,

$$P = \frac{1}{3V}\int_0^\infty pE'(p)\bar{n}(p)dp\,, \qquad (*)$$

in the case of a massless particle $E = pc$ with $\mu = 0$ to find

$$P = \begin{cases} \frac{2\sigma}{3c} g_s T^4\,, & \text{(bosons)} \\ \frac{7}{8}\frac{2\sigma}{3c} g_s T^4\,, & \text{(fermions)} \end{cases}$$

where $\sigma = \pi^2 k^4/60\hbar^3 c^2$ is the Stefan-Boltzmann constant. [*Hint:* Make the substitution $z = \beta E$ and note that the Reimann zeta function $\zeta(n+1) = \frac{1}{n!}\int_0^\infty \frac{z^n}{e^z - 1}dz$ with $\zeta(4) = \pi^4/90$.]