EXAMPLES IV

Spherical Collapse of Radiation Suppose we wanted to get an idea of how density perturbations evolved during the radiation era. We could try to naively repeat our arguments regarding spherical collapse of dark matter but with radiation. If we make a overdensity of radiation by compressing a sphere from $\overline{R}$ to R then we can make the same argument that the two regions will be gravitationally decoupled with the outside being described by

$$\left(\frac{\dot{\overline{a}}}{\overline{a}}\right)^2 = \frac{H_0^2\overline{\Omega}_{r,0}}{\overline{a}^4} = \frac{H_0^2}{\overline{a}^4}$$

and the inside will be described by

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{H_0^2\Omega_{r,0}}{a^4} - \frac{kc^2}{a^2}$$

Show that these have solutions

$$\overline{a} = \sqrt{2H_0 t}$$
$$a = A\sin(\theta)$$
$$t = B(1-\cos\theta)$$

where you should determine A, B and θ. Now expand the solution for a in terms of θ to show that at zeroth order

$$R = \overline{R}$$

as we expect (you will need that $\Omega_{r,0}^{1/4} = \overline{R}_0/R_0$ as the density of radiation goes as a^4 rather than a^3 as for matter). Show that at first order the linear density perturbation grow as

$$\delta_{linear} = \frac{\Omega_{r,0}-1}{2\sqrt{\Omega_{r,0}}}a^2 \propto t$$

The CMB was emitted at $t_{CMB} \approx 4\times 10^5$ years and we know that the perturbations then were $\delta \approx 10^{-5}$ so can be described by linear theory. If we take the beginning of the radiation era to be at the end of inflation $t \approx 10^{-32}$s we see that our growth of perturbations should be huge so our initial perturbations must have been very tiny. What is wrong with the argument we have made? (In reality the perturbations during the radiation era and don't grow in amplitude by much at all)

General Virial Theorem Consider a general force law where $F = \nabla V$ and

$$V = \alpha r^n$$

By following the derivation in your notes derive the virial theorem for this force law:

$$nV = 2K$$

Now let us consider relativistic particles. Using special relativity we know that the kinetic energy, K, of a particle changes from:

$$\frac{1}{2}mv^2 \to (\gamma-1)mc^2 \qquad \left(\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right)$$

where m is the rest mass. Show that the term we previously labeled the twice kinetic energy, $\sum_i m_i v^2 = 2K$, in our derivation of the virial theorem is now:

$$\frac{\gamma+1}{\gamma}K$$

(you will need to remember that the mass in our derivation will be the relativistic mass, not the rest mass) and so the general virial theorem will become:

$$nV = \frac{\gamma+1}{\gamma}K$$

By considering total energy discuss the ranges of $\{n, \gamma\}$ for which stable bound systems exist. In particular derive the result that an ultra-relativistic star with Newtonian gravity is only marginally stable (as we found using pressure support in class).

Halo formation One consequence of the virial theorem is that we can estimate the time at which halos formed from their mass M and their velocity dispersion σ_v. We know from our discussion of spherical collapse that

$$\rho_{vir} \approx 200\bar{\rho}(t_{vir})\,. \qquad (*)$$

Use the kinetic energy ($K = \frac{1}{2}M\sigma_v$) and potential energy ($V = -GM^2/R_{vir}$) of the halo with the virial theorem to show,

$$R_{vir} = \frac{GM}{\sigma_v^2}\,.$$

Now write down the background density at the time of virialisation, $\bar{\rho}(t_{vir})$, in terms of $\Omega_{m,0}$ and z_{vir} and the halo density ρ_{vir} in terms of M and R_{vir}. Then use (*) to show,

$$1 + z_{vir} \approx \left(100G^2H_0^2\Omega_{m,0}\right)^{-\frac{1}{3}}\left(\frac{\sigma_v}{M^{\frac{1}{3}}}\right)^2\,.$$

This tells us that low-mass high-velocity objects virialised first and high-mass low-velocity objects virialised last. For our galaxy we have $\sigma_v \approx 300kms^{-1}$ and $M \approx 10^{12}M_{\text{sun}}$ so we find that $z_{vir} \leq 7$. The general result is that it is hard to form anything before $z \approx 10$, which defines the cosmological dark ages (no galaxies).

Polytropes Solve the Lane-Emden equation,

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right] = -\theta^n\,,$$

for $n = 0$ and confirm the solutions for $n = 1$ and $n = 5$,

$$\theta_1 = \frac{\sin\xi}{\xi} \quad \text{and} \quad \theta_5 = 1/\sqrt{1+\frac{1}{3}\xi^2}\,.$$

via substitution.

Pressure support equation Let r be the radial distance from the centre of a spherically symmetric star of pressure $P(r)$, and let $m(r)$ be the mass within a sphere of radius r. Use the pressure-support equations to show that the function

$$F(r) = P + \frac{Gm^2}{8\pi r^4}$$

is a decreasing function of r. Let M be the mass of the star and R its radius. Derive the lower bound

$$P_c > GM^2/(8\pi R^4)$$

on the central pressure P_c.

Thermostatic equilibrium A star is assumed to be a spherically-symmetric ball of ideal gas held together by gravity. Assuming that the number density $n(r)$, pressure $P(r)$ and temperature $T(r)$ are functions only of radial distance r from the centre, use the ideal gas law (Boyle-Charles law) to show that their gradients n', P' and T' are related by

$$\frac{n'}{n} = \frac{P'}{P} - \frac{T'}{T}$$

**Please send any corrections to jf334@damtp.cam.ac.uk*