Further Complex Methods, Examples sheet 2 Prof. M. J. Perry

Comments and corrections: e-mail to mjp1@cam.ac.uk.

1 By using a contour consisting of the boundary of a quadrant, indented at the origin, show that (for a range of z to be stated)

$$\int_0^\infty t^{z-1}e^{-it}\,dt = e^{-\frac{1}{2}\pi i z}\Gamma(z).$$

Hence evaluate (again, for ranges of z to be stated)

$$\int_0^\infty t^{z-1}\cos t\,dt \quad \text{and} \quad \int_0^\infty t^{z-1}\sin t\,dt.$$

Use your results to evaluate $\displaystyle\int_0^\infty \frac{\cos t}{t^{1/2}}\,dt$, $\displaystyle\int_0^\infty \frac{\sin t}{t}\,dt$ and $\displaystyle\int_0^\infty \frac{\sin t}{t^{3/2}}\,dt$.

2 Derive the formula $\mathrm{B}(p,q) = 2\int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta\cos^{2q-1}\theta\,d\theta$ and prove that

$$\mathrm{B}(z,z) = 2^{1-2z}\mathrm{B}(z,\tfrac{1}{2}).$$

For which values of z does this result hold?

3 Show, using properties of the B function, that

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} = \tfrac{1}{\sqrt{32\pi}}\left(\Gamma(\tfrac{1}{4})\right)^2$$

Using the change of variable $x = t(2-t^2)^{-\frac{1}{2}}$, deduce that

$$\mathrm{K}(\tfrac{1}{\sqrt{2}}) = \tfrac{4}{\sqrt{\pi}}\left(\Gamma(\tfrac{5}{4})\right)^2 ,$$

where $\mathrm{K}(k)$ is the complete elliptic integral $\displaystyle\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}$.

4 By using the infinite product representation of the Gamma function (Weierstrass canonical product), prove that

$$\frac{2^{2z}\Gamma(z)\Gamma(z+\frac{1}{2})}{\Gamma(2z)}$$

is a constant independent of z. Then, by letting $z \to 0$ evaluate the relevant constant and thus establish the following identity:

$$2^{2z-1}\Gamma(z)\Gamma(z+\frac{1}{2}) = \sqrt{\pi}\Gamma(2z).$$

Furthermore, show that for $m = 1,2,3,\ldots$, the following identity is valid:

$$\Gamma(z)\Gamma(z+\frac{1}{m})\Gamma(z+\frac{2}{m})\ldots\Gamma(z+\frac{m-1}{m}) = m^{\frac{1}{2}-mz}(2\pi)^{\frac{m-1}{2}}\Gamma(mz).$$

5 Using $t = s\tau$, $s > 0$, it follows that

$$\frac{\Gamma(z)}{s^z} = \int_0^\infty e^{-s\tau}\tau^{z-1}d\tau.$$

Letting $z = 1$ and integrating the resulting formula with respect to s from 1 to t, show that

$$\ln t = \int_0^\infty \left(e^{-\tau} - e^{-t\tau}\right)\frac{d\tau}{\tau}.$$

Using this formula in the expression for $\Gamma'(z)$, prove that

$$\frac{\Gamma'(z)}{\Gamma(z)} = \int_0^\infty \left(e^{-\tau} - \frac{1}{(1+\tau)^z}\right)\frac{d\tau}{\tau}.$$

Hence, deduce that

$$\gamma = -\int_0^\infty \left(e^{-\tau} - \frac{1}{1+\tau}\right)\frac{d\tau}{\tau}\ .$$

6 Starting with the infinite product representation of the Gamma function and using the definition of γ, derive Euler's product representation, i.e.

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\ n^z}{z(1+z)(2+z)\dots(n+z)}\ .$$

7 (a) Prove that for Re$z > 1$,

$$\frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t-1}\,dt = \frac{\Gamma(1-z)}{2i\pi}\int_\gamma \frac{t^{z-1}}{e^{-t}-1}\,dt,$$

where γ denotes the Hankel contour. Hence, deduce that the RHS of the above equation provides the analytic continuation of Riemann's zeta function.

(b) The Bernoulli numbers B_n are defined by

$$\frac{1}{e^t-1} = \sum_{m=0}^\infty B_m \frac{t^{m-1}}{m!},$$

and $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_{2m+1} = 0$ for $m = 1, 2, \dots$.

Use (a) and the residue theorem to compute $\zeta(-n)$, $n = 0, 1, 2, \dots$ in terms of B_n. Hence, deduce that the negative even integers are zeros of $\zeta(z)$.

8 Show that

$$E_1(k) = \int_k^\infty \frac{e^{-t}}{t}\,dt = -\gamma - \ln k + k - \frac{k^2}{4} + O(k^3), \quad k \to 0^+.$$

Hint:

$$E_1(k) = \int_k^\infty \frac{dt}{t(t+1)} + \int_0^\infty \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t} - \int_0^k \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t}.$$

9

Show that for $Re\, z > 0$

$$(1-2^{1-z})\zeta(z) = (1^{-z} - 2^{-z} + 3^{-z} - 4^{-z} \cdots) = \frac{1}{\Gamma(z)} \int_0^\infty \frac{t^{z-1}}{e^t + 1} dt.$$

.

10

Show that

$$\int_{-\infty}^{(0+)} \frac{\ln t}{e^{-t} - 1} dt = 0.$$

Hence show that

$$\lim_{z \to 1} (\zeta(z) - (z-1)^{-1}) = \gamma,$$

and

$$\zeta'(0) = -\ln \sqrt{2\pi}.$$

11

The psi-function is defined to be

$$\psi(z) = \frac{d}{dz} \ln \Gamma(z).$$

Show that

$$\psi'(z) = \sum_{s=0}^{\infty} \frac{1}{(s+z)^2}, \qquad (z \neq 0, -1, -2 \cdots).$$

Then show that when z is real and positive, that $\Gamma(z)$ has a single minimum which lies somewhere between $z = 1$ and $z = 2$.

Hence show that

$$\ln \Gamma(z) = -\gamma(z-1) + \sum_{s=2}^{\infty} (-)^s \frac{\zeta(s)}{s} (z-1)^s.$$