Further Complex Methods, Examples sheet 3 Prof. M. J. Perry

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**1** Let $\omega_{m,n} = m\omega_1 + n\omega_2$, where $(m, n)$ are integers not both zero, and let

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{m,n}^{\infty} \frac{1}{(z-\omega_{m,n})^2} - \frac{1}{\omega_{m,n}{}^2}$$

be the Weierstrass elliptic function with periods $(\omega_1, \omega_2)$ such that $\omega_1/\omega_2$ is not real. Show that, in a neighbourhood of $z = 0$,

$$\mathcal{P}(z) = \frac{1}{z^2} + \frac{1}{20} g_2 z^2 + \frac{1}{28} g_3 z^4 + O(z^6)$$

where

$$g_2 = 60 \sum_{m,n} (\omega_{m,n})^{-4}, \quad g_3 = 140 \sum_{m,n} (\omega_{m,n})^{-6}.$$

Deduce that $\mathcal{P}$ satisfies a 1st order nonlinear ODE

$$(\mathcal{P}')^2 = 4\mathcal{P}^3 - g_2\mathcal{P} - g_3.$$

**2** Show that

$$4\mathcal{P}(2z) - \left(\frac{\mathcal{P}''(z)}{\mathcal{P}'(z)}\right)^2 + 8\mathcal{P}(z) = 0.$$

**3** The function $\sin^{-1} z$ is defined, for $0 \le \arg z < 2\pi$, by

$$\sin^{-1} z = \int_0^z \frac{dt}{\sqrt{1-t^2}} \,,$$

where the integrand has a branch cut along the real axis from $-1$ to $+1$ and takes the value $+1$ at the origin on the upper side of the cut. The path of integration is a straight line for $0 \le \arg(z) \le \pi$ and is curved in a positive sense round the branch cut for $\pi < \arg z < 2\pi$. Express $\sin^{-1}(e^{i\pi} z)$ $(0 < \arg z < \pi)$ in terms of $\sin^{-1} z$ and deduce that $\sin(\phi - \pi) = -\sin\phi$. *Hint:* $\sin^{-1}(e^{i\pi} z) = -\pi + \sin^{-1} z$, as can be derived by calculating the integral half way round the cut and remembering that the integrand is an odd function.

**4** Let

$$F(z) = \int_{-\infty}^{\infty} \frac{e^{uz}}{1+e^u}\, du \,.$$

For what region of the $z$-plane does $F(z)$ define an analytic function?

Show by closing the contour (use a rectangle) in the upper half plane that

$$F(z) = \pi \operatorname{cosec} \pi z \,.$$

Explain how this result provides the analytic continuation of $F(z)$.

**5** Find two independent solutions of the Airy equation $w'' - zw = 0$ in the form

$$w(z) = \int_\gamma e^{zt} f(t)\, dt,$$

where $\gamma$ is to be specified in each case. Show that there is a solution for which $\gamma$ can be chosen to consist of two straight line segments in the left half $t$-plane (Re $t \le 0$).

For this solution show that, if $w(z)$ is normalised so that $w(0) = iA\, 3^{-\frac{1}{6}}\Gamma(1/3)$, where $A$ is a constant, then $w'(0) = -iA\, 3^{\frac{1}{6}}\Gamma(2/3)$.

[Note: $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$ for $\Re z > 0$.]

**6** By writing $w(z)$ in the form of an integral representation with the Laplace kernel show that the confluent hypergeometric equation $zw'' + (c - z)w' - aw = 0$ has solutions of the form

$$w(z) = \int_\gamma t^{a-1}(1-t)^{c-a-1} e^{tz}\, dt,$$

provided the path $\gamma$ is chosen such that $[t^a(1-t)^{c-a}e^{tz}]_\gamma = 0$.

In the case Re $z > 0$, find paths which provide two independent solutions in each of the following cases (where $m$ is a positive integer):
(i) $a = -m$, $c = 0$;
(ii) Re $a < 0$, $c = 0$, $a$ is not an integer;
(iii) $a = 0$, $c = m$;
(iv) Re $c >$ Re $a > 0$, $a$ and $c - a$ are not integers.

**7** Use the Laplace transform to solve the ordinary differential equation

$$\frac{d^2y}{dt^2} - k^2 y = f(t), \quad k > 0, \quad y(0) = y_0, \quad y'(0) = y_0'.$$

Let $f(t) = e^{-k_0 t}, k_0 \ne k, k_0 > 0$, so that the Laplace transform of $f(t)$ is

$$\hat{f}(s) = \frac{1}{s + k_0}.$$

Show that

$$y(t) = y_0 \cosh kt + \frac{y_0'}{k} \sinh kt + \frac{e^{-k_0 t}}{k_0^2 - k^2} - \frac{\cosh kt}{k_0^2 - k^2} + \frac{\frac{k_0}{k}}{k_0^2 - k^2} \sinh kt.$$

Now suppose that $f(t)$ is an arbitrary continuous function that possesses a Laplace transform. Use the convolution theorem for Laplace transforms, or otherwise, to show that

$$y(t) = y_0 \cosh kt + \frac{y_0'}{k} \sinh kt + \int_0^t f(t') \frac{\sinh k(t - t')}{k} dt'.$$

Put $f(t) = e^{-k_0 t}$ and re-obtain your answer to the first part of this question. Suppose now that $k_0 = k$. What is $f(t)$? Could you have found this solution by taking the limit in $*$ as $k_0 \to k$?

**8**

The Schrödinger equation is

$$i\frac{\partial u}{\partial t} + \frac{\partial^2 u}{\partial x^2} = 0.$$

Suppose that $u(x,0) = f(x)$.
Fourier transform this equation with respect to $x$ to find

$$u(x,t) = \frac{e^{-i\pi/4}}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{\frac{i(x-x')^2}{4t}} f(x')dx'.$$

(You may it useful to recall that $\int_{-\infty}^{\infty} e^{iu^2} du = e^{\frac{i\pi}{4}}\sqrt{\pi}$.)

Now use Laplace transform methods to find the same solution to this problem.