Further Complex Methods, Revision Examples sheet Prof. A.S. Fokas

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Short questions

1 Starting from the Euler integral and using the power series expansion for e^{-t}, show that, for $\Re z > 0$,

$$\Gamma(z) = \sum_0^\infty \frac{(-1)^n}{(z+n)n!} + \int_1^\infty e^{-t}t^{z-1}dt \; .$$

Explain briefly how result this provides the analytic continuation of the Euler integral, and use it to evaluate the residue at $z = -m$.

2 Show that the most general linear second order ordinary differential equation whose only singularities are regular singular points at $z = a$ and $z = b$ can be written in the form

$$w'' + \left[\frac{1-A}{z-a} + \frac{1+A}{z-b}\right] w' + \frac{B(a-b)^2}{(z-a)^2(z-b)^2} w \;=\; 0 \, , \qquad (\dagger)$$

where A and B are arbitrary constants.

Write down and solve the equation when the two singular points are at 0 and ∞, in the two cases $A^2 \neq 4B$ and $A^2 = 4B$.

3 Show, using the change of variable $x = 2t - 1$ and any standard properties of special functions you like, that

$$\int_{-1}^1 \frac{dx}{(1+x)^{2/3}(1-x)^{1/3}} = \frac{2\pi}{\sqrt{3}} \, .$$

4 Find the analytic function whose real part is given by

$$u(x,y) = \frac{x + \sin\alpha}{x^2 + y^2 + 1 + 2(x\sin\alpha + y\cos\alpha)}, \quad x, y \in \mathbb{R}$$

where α is a real constant.

5 The function $\mathrm{Tan}^{-1}\, z$ is defined by

$$\mathrm{Tan}^{-1}\, z = \int_0^z \frac{dt}{1+t^2} \, . \qquad (*)$$

where the path of integration is a straight line. For what region of the complex z plane is $\mathrm{Tan}^{-1}\, z$ analytic?

The function $\tau\mathrm{an}^{-1} z$ is defined by the same integral as $(*)$, except that the path of integration consists of a line segment from $t = 0$ to $t = 1$, then a line segment from $t = 1$ to $t = z$. Show that $\tau\mathrm{an}^{-1} z$ is an analytic continuation of $\mathrm{Tan}^{-1}\, z$ and determine the nature of the singular points of these functions.

Show (without assuming any properties of $\tan^{-1}$ or log) that the possible values of the functions obtained by analytic continuation of $\mathrm{Tan}^{-1}\, z$ are

$$\tan^{-1} z = \mathrm{Tan}^{-1}\, z + m\pi$$

where m is any integer.

6 Define $E_1(k)$ by

$$E_1(k) = \int_k^\infty \frac{e^{-t}}{t} dt, \quad k > 0.$$

This integral can be rewritten in the form

$$E_1(k) = \int_k^\infty \frac{dt}{t(t+1)} + \int_0^\infty \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t} - \int_0^k \left(e^{-t} - \frac{1}{t+1}\right)\frac{dt}{t}. \tag{1}$$

The first integral can be computed explicitly and the second integral equals $-\gamma$ (see example sheet 3, problem 4). By estimating the third integral of the RHS of (1), show that

$$E_1(k) = -\gamma - \ln k + k - \frac{k^2}{4} + O\left(k^3\right), \quad k \to 0^+.$$

Long Questions

7 Show carefully, by integrating $\dfrac{1}{(1+z)^{2/3}(z-1)^{1/3}}$ round a large circle, that

$$\int_{-1}^{1} \frac{dx}{(1+x)^{2/3}(1-x)^{1/3}} = \frac{2\pi}{\sqrt{3}}\ .$$

8 Solve the following singular integral equation:

$$\varphi(x) + \frac{\alpha}{i\pi} \fint_{-\infty}^{\infty} \frac{\varphi(\xi)}{\xi - x} d\xi = \frac{\sin x}{x}, \quad x \in \mathbb{R},$$

where α is a constant different than ± 1 and $\fint$ denotes the principal value integral.

9 Consider the following initial-boundary value problem:

$$\begin{aligned} iq_t + q_{xx} &= 0, \quad 0 < x < \infty,\ t > 0 \\ q(x,0) &= e^{-x}, \quad 0 < x < \infty, \\ q(0,t) &= \cos t, \quad t > 0, \end{aligned}$$

where q vanishes for all t as $x \to \infty$.

1. Express the solution as an integral in the complex k-plane.
2. Use appropriate contour deformations so that the relevant integrand decays exponentially as $|k| \to \infty$.

10 Show that functions of the form

$$w(z) = \int_\gamma f(t) e^{zt} dt$$

satisfy the equation

$$zw''' + (2-a)w'' - zw' + aw = 0,$$

provided that $f(t) = t^{-a-1}$. Taking $\Re z > 0$, sketch paths that give linearly independent solutions in the cases: (i) a is not an integer; (ii) a is a negative integer. You need not justify the independence of your solutions.