Classical Dynamics: Example Sheet 3

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1. Tensor of Inertia

a. Prove that the principal moments of inertia, I_a, are real and non-negative.

b. During the lectures we have outlined the proof of the Parallel Axis Theorem, which is a statement that the inertia tensor about a point P', which is displaced by $\mathbf{c}$ from the centre of mass is related to the inertia tensor about the centre of mass as

$$(I_{P'})_{ab} = (I_{CoM})_{ab} + M(c^2\delta_{ab} - \mathbf{c}_a\mathbf{c}_b), \tag{1}$$

where M is the total mass of the body. Complete the proof of the theorem. Hint: it will be helpful to choose the origin in the centre of mass.

2. Euler's Angles

Show that the effect of three rotations by Euler angles results in the relationship $\mathbf{e}_a = R_{ab}\tilde{\mathbf{e}}_b$ between the body frame axes $\{\mathbf{e}_a\}$ and the space frame axes $\{\tilde{\mathbf{e}}_b\}$ where the orthogonal matrix R is

$$R = \begin{pmatrix} \cos\psi\cos\phi - \cos\theta\sin\phi\sin\psi & -\cos\phi\sin\psi - \cos\theta\cos\psi\sin\phi & \sin\theta\sin\phi \\ \sin\phi\cos\psi + \cos\theta\sin\psi\cos\phi & -\sin\psi\sin\phi + \cos\theta\cos\psi\cos\phi & -\sin\theta\cos\phi \\ \sin\theta\sin\psi & \sin\theta\cos\psi & \cos\theta \end{pmatrix}^T$$

[Hint: mind the order in which the individual rotational matrices are multiplied!]

Use this to find the angular velocity $\boldsymbol{\omega}$ expressed in terms of Euler angles in (a) the body frame, and (b) the space frame.

3. Free Symmetric Top

(a) Consider a torque-free motion of a round plate. Show that in the body frame the vector of the angular velocity $\boldsymbol{\omega}$ precesses about the body axis $\boldsymbol{e}_3$ with constant angular frequency equal to ω_3.

(b) The physicist Richard Feynman tells the following story:

> "*I was in the cafeteria and some guy, fooling around, throws a plate in the air. As the plate went up in the air I saw it wobble, and I noticed the red medallion of Cornell on the plate going around. It was pretty obvious to me that the medallion went around faster than the wobbling.*
>
> *I had nothing to do, so I start figuring out the motion of the rotating plate. I discover that when the angle is very slight, the medallion rotates twice as fast as the wobble rate – two to one. It came out of a complicated equation!*
>
> *I went on to work out equations for wobbles. Then I thought about how the electron orbits start to move in relativity. Then there's the Dirac equation in electrodynamics. And then quantum electrodynamics. And before I knew it....the whole business that I got the Nobel prize for came from that piddling around with the wobbling plate.*"

[Here the "wobble" is associated with precession of the top about $\tilde{\boldsymbol{e}}_3$ (motion in ϕ), not nutation (motion in θ).] Feynman was right about quantum electrodynamics. But what about the plate?
[You could try two alternative methods. One, by using the expression for ω_3 in terms of Euler's angles together with the expression for Ω – the angular frequency of precession of $\boldsymbol{\omega}$ about $\boldsymbol{e}_3$, derived in lectures. Second, by writing down the Lagrangian of the top and deriving the equation of motion for the θ-component.]

(c) Consider a uniform symmetric ellipsoid of mass M with $a = b \neq c$ (see example sheet 2, q. 5). Find the ratio of the semi-axes, for which $\dot{\phi}$ – the angular frequency of precession of the top about the vector of angular momentum $\boldsymbol{L}$ – equals $\omega_3/(5\cos\theta)$. Deduce further that the spin of the top is $\dot{\psi} = \frac{4}{5}\omega_3$. What is the relationship between $\dot{\phi}$ and $\dot{\psi}$ for small values of θ? Compare with the result obtained in (b).

4. Free Asymmetric Top (1)

(a) Throw a book in the air. If the principal moments of inertia are $I_1 > I_2 > I_3$, convince yourself that the book can rotate in a stable manner about the principal axes $\mathbf{e}_1$ and $\mathbf{e}_3$, but not about $\mathbf{e}_2$.

(b) Use Euler's equations to show that the energy E and the total angular momentum $\mathbf{L}^2$ of a free asymmetric top are conserved. Suppose that the initial conditions are such that

$$\mathbf{L}^2 = 2I_2E \tag{2}$$

with the initial angular velocity $\boldsymbol{\omega}$ perpendicular to the intermediate principal axes $\mathbf{e}_2$. Show that $\boldsymbol{\omega}$ will ultimately end up parallel to $\mathbf{e}_2$ and derive the characteristic time taken to reach this steady state.

5. Free Asymmetric Top (2)

A rigid lamina (i.e. a two dimensional object) has principal moments of inertia about the centre of mass given by,

$$I_1 = (\mu^2 - 1) \qquad I_2 = (\mu^2 + 1) \quad , \quad I_3 = 2\mu^2 \tag{3}$$

Write down Euler's equations for the lamina moving freely in space. Show that the component of the angular velocity in the plane of the lamina (i.e. $\sqrt{\omega_1^2 + \omega_2^2}$) is constant in time.

Choose the initial angular velocity to be $\boldsymbol{\omega} = \mu N \mathbf{e}_1 + N \mathbf{e}_3$. Define $\tan\alpha = \omega_2/\omega_1$, which is the angle the component of $\boldsymbol{\omega}$ in the plane of the lamina makes with $\mathbf{e}_1$. Show that it satisfies

$$\ddot{\alpha} + N^2 \cos\alpha \sin\alpha = 0 \tag{4}$$

and deduce that at time t,

$$\boldsymbol{\omega} = [\mu N \operatorname{sech} Nt]\mathbf{e}_1 + [\mu N \tanh Nt]\mathbf{e}_2 + [N \operatorname{sech} Nt]\mathbf{e}_3 \tag{5}$$

6. Heavy Symmetric Top

Consider a heavy symmetric top of mass M, pinned at point P which is a distance l from the centre of mass (see Figure 1). The principal moments of inertia about P are I_1, I_1 and I_3 and the Euler angles are shown in the figure. The top is spun with initial conditions $\dot{\phi} = 0$ and $\theta = \theta_0$. Show that θ obeys the equation of motion,

$$I_1 \ddot{\theta} = -\frac{dV_{\text{eff}}(\theta)}{d\theta} \tag{6}$$

where

$$V_{\text{eff}}(\theta) = \frac{I_3^2 \omega_3^2}{2I_1} \frac{(\cos\theta - \cos\theta_0)^2}{\sin^2\theta} + Mgl\cos\theta \tag{7}$$

Suppose that the top is spinning very fast so that

$$I_3 \omega_3 \gg \sqrt{MglI_1} \tag{8}$$

Show that θ_0 is close to the minimum of $V_{\text{eff}}(\theta)$. Use this fact to deduce that the top nutates with frequency

$$\Omega \approx \frac{\omega_3 I_3}{I_1} \tag{9}$$

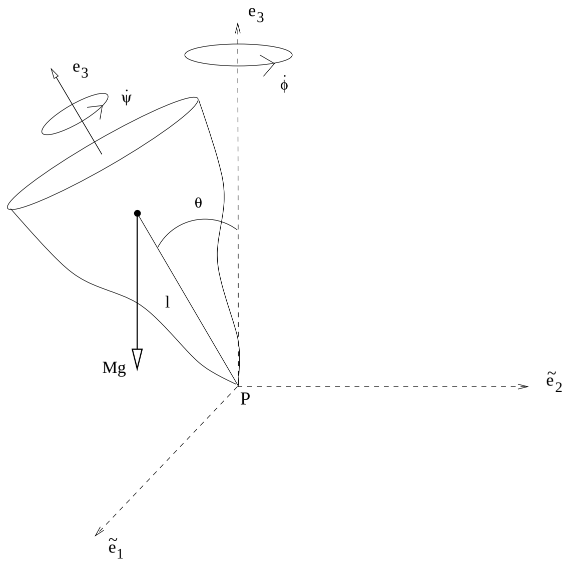

Figure 1: The Euler angles for the heavy symmetric top

and draw the subsequent motion.

7. Heavy Symmetric Top in Hamiltonian Formalism

The Lagrangian for the heavy symmetric top is

$$L = \frac{1}{2}I_1\left(\dot{\theta}^2 + \dot{\phi}^2\sin^2\theta\right) + \frac{1}{2}I_3(\dot{\psi} + \dot{\phi}\cos\theta)^2 - Mgl\cos\theta \tag{10}$$

Obtain the momenta p_θ, p_ϕ and p_ψ and the Hamiltonian $H(\theta, \phi, \psi, p_\theta, p_\phi, p_\psi)$. Derive Hamilton's equations.

8. Hamilton's Equations

A system with two degrees of freedom x and y has the Lagrangian,

$$L = x\dot{y} + y\dot{x}^2 + \dot{x}\dot{y} \tag{11}$$

Derive Lagrange's equations. Obtain the Hamiltonian $H(x, y, p_x, p_y)$. Derive Hamilton's equations and show that they are equivalent to Lagrange's equations.