Partial Differential Equations

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Books

In addition to the sets of lecture notes written by previous lecturers ([1, 2]) which are still useful, the books [4, 3] are very good for the PDE topics in the course, and go well beyond the course also. If you want to read more on distributions [6] is most relevant. Also [7, 8] are useful; the books [5, 9] are more advanced, but may be helpful.

References

[1] T.W. Körner, Cambridge Lecture notes on PDE, available at https://www.dpmms.cam.ac.uk/ twk

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[3] G.B. Folland, Introduction to Partial Differential Equations, *Princeton 1995, QA 374 F6*

[4] L.C. Evans, Partial Differential Equations,*AMS Graduate Studies in Mathematics Vol 19, QA377.E93 1990*

[5] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, *Springer, New York 2011 QA320 .B74 2011*

[6] F.G. Friedlander, Introduction to the Theory of Distributions, *CUP 1982, QA324*

[7] F. John, Partial Differential Equations, *Springer-Verlag 1982, QA1.A647*

[8] Rafael Jos Iorio and Valria de Magalhes Iorio, Fourier analysis and partial differential equations *CUP 2001, QA403.5 .I57 2001*

[9] M.E. Taylor, Partial Differential Equations,Vols I-III*Springer 96, QA1.A647*

1 Introduction

1.1 Notation

We write partial derivatives as $\partial_t = \frac{\partial}{\partial t}, \partial_j = \frac{\partial}{\partial x_j}$ etc and also use suffix on a function to indicate partial differentiation: $u_t = \partial_t u$ etc. A general k^{th} order linear partial differential operator (pdo) acting on functions $u = u(x_1, \ldots x_n)$ is written:

$$P = \sum_{|\alpha \le k} a_\alpha \partial^\alpha u\,. \tag{1.1.1}$$

Here $\alpha = (\alpha_1, \ldots \alpha_n) \in \mathbb{Z}_+^n$ is a multi-index of order $|\alpha| = \sum \alpha_j$ and

$$\partial^\alpha = \prod \partial_j^{\alpha_j}\,, \quad x^\alpha = \prod x_j^{\alpha_j}\,. \tag{1.1.2}$$

For a multi-index we define the factorial $\alpha! = \prod \alpha_j!$. For (real or complex) constants a_α the formula (1.1.1) defines a constant coefficient linear pdo of order k. (Of course assume always that at least one of the a_α with $|\alpha| = k$ is non-zero so that it is genuinely of order k.) If the coefficients depend on x it is a variable coefficient linear pdo. The word linear means that

$$P(c_1 u_1 + c_2 u_2) = c_1 P u_1 + c_2 P u_2 \tag{1.1.3}$$

holds for P applied to C^k functions u_1, u_2 and arbitrary constants c_1, c_2.

1.2 Basic definitions

If the coefficients depend on the partial derivatives of a function of order strictly less than k the operator

$$u \mapsto Pu = \sum_{|\alpha \le k} a_\alpha(x, \{\partial^\beta u\}_{|\beta|<k})\, \partial^\alpha u \tag{1.2.1}$$

is called quasi-linear and (1.1.3) no longer holds. The corresponding equation $Pu = f$ for $f = f(x)$ is a quasi-linear partial differential equation (pde). In such equations the partial derivatives of highest order - which are often most important - occur linearly. If the coefficients of the partial derivatives of highest order in a quasi-linear operator P depend only on x (not on u or its derivatives) the equation is called semi-linear. If the partial derivatives of highest order appear nonlinearly the equation is called fully nonlinear; such a general pde of order k may be written

$$F(x, \{\partial^\alpha u\}_{|\alpha| \le k}) = 0\,. \tag{1.2.2}$$

Definition 1.2.1 *A classical solution of the pde* (1.2.2) *on an open set* $\Omega \subset \mathbb{R}^n$ *is a function* $u \in C^k(\Omega)$ *which is such that* $F(x, \{\partial^\alpha u(x)\}_{|\alpha| \le k}) = 0$ *for all* $x \in \Omega$ *.*

Classical solutions do not always exist and we will define generalized solutions later in the course. The most general existence theorem for classical solutions is the Cauchy-Kovalevskaya theorem, to state which we need the following definitions:

Definition 1.2.2 *Given an operator* (1.1.1) *we define*

- $P_{principal} = \sum_{|\alpha=k} a_\alpha \partial^\alpha u$, *(principal part)*
- $p = \sum_{|\alpha \le k} a_\alpha (i\xi)^\alpha$, $\quad \xi \in \mathbb{R}^n$, *(total symbol)*
- $\sigma = \sum_{|\alpha=k} a_\alpha (i\xi)^\alpha$, $\quad \xi \in \mathbb{R}^n$, *(principal symbol)*
- $Char_x(P) = \{\xi \in \mathbb{R}^n : \sigma(x,\xi) = 0\}$, *(the set of characteristic vectors at x)*
- $Char(P) = \{(x,\xi) : \sigma(x,\xi) = 0\} = \cup_x Char_x(P)$, *(characteristic variety)*.

Clearly σ, p depend on $(x,\xi) \in \mathbb{R}^{2n}$ for variable coefficient linear operators, but are independent of x in the constant coefficient case. For quasi-linear operators we make these definitions by substituting in $u(x)$ into the coefficients, so that p, σ and (also the definition of characteristic vector) depend on this $u(x)$.

Definition 1.2.3 *The operator* (1.1.1) *is elliptic at x (resp. everywhere) if the principal symbol is non-zero for non-zero ξ at x (resp. everywhere). (Again the definition of ellipticity in the quasi-linear case depends upon the function $u(x)$ in the coefficients.)*

The elliptic operators are an important class of operators, and there is a well-developed theory for elliptic equations $Pu = f$. Other important classes of operators are the parabolic and hyperbolic operators: see the introductions to sections 4 and 5 for defintions of classes of parabolic and hyperbolic operators of second order.

1.3 The Cauchy-Kovalevskaya theorem

The *Cauchy problem* is the problem of showing that for a given pde and given data on a hypersurface $\mathcal{S} \subset \mathbb{R}^n$ there is a unique solution of the pde which agrees with the data on $\mathcal{S}$. This is a generalization of the initial value problem for ordinary differential equations, and by analogy the appropriate data to be given on $\mathcal{S}$ consists of u and its normal derivatives up to order $k-1$. A crucial condition is the following:

Definition 1.3.1 *A hypersurface $\mathcal{S}$ is non-characteristic at a point x if its normal vector $n(x)$ is non-characteristic, i.e. $\sigma(x, n(x)) \neq 0$. We say that $\mathcal{S}$ is non-characteristic if it is non-characteristic for all $x \in \mathcal{S}$.)*

Again for quasi-linear operators it is necessary to substitute $u(x)$ to make sense of this definition, so that whether or not a hypersurface is non-characteristic depends on $u(x)$, which amounts to saying it depends on the data which are given on $\mathcal{S}$.

Theorem 1.3.2 (Cauchy-Kovalevskaya theorem) *In the real analytic case there is a local solution to the Cauchy problem for a quasi-linear pde in a neighbourhood of a point as long as the hypersurface is non-characteristic at that point.*

This becomes clearer with a suitable choice of coordinates which emphasizes the analogy with ordinary differential equations: let the hypersurface be the level set $x_n = t = 0$ and let $x = (x_1, \dots x_{n-1})$ be the remaining $n-1$ coordinates. Then a quasi-linear P takes the form

$$Pu = a_{0k}\partial_t^k + \sum_{|\alpha|+j\le k, j<k} a_{j\alpha}\partial_t^j \partial^\alpha u \tag{1.3.1}$$

with the coefficients depending on derivatives of order $< k$, as well as on (x, t). Since the normal vector to $t = 0$ is $n = (0, 0, \dots 0, 1) \in \mathbb{R}^n$ the non-characteristic condition is just $a_{0k} \neq 0$, and ensures that the quasi-linear equation $Pu = f$ can be solved for $\partial_t^k u$ in terms of $\{\partial_t^j \partial^\alpha u\}_{|\alpha|+j\le k, j<k}$ to yield an equation of the form:

$$\partial_t^k u = G(x, t, \{\partial_t^j \partial^\alpha u\}_{|\alpha|+j\le k, j<k}) \tag{1.3.2}$$

to be solved with data

$$u(x, 0) = \phi_0(x), \partial_t u(x, 0) = \phi_1(x) \dots \partial_t^{k-1} u(x, 0) = \phi_{k-1}(x)\,. \tag{1.3.3}$$

Notice that these data determine, for all $j < k$, the derivatives

$$\partial_t^j \partial^\alpha u(x, 0) = \partial^\alpha \phi_j(x)\,, \tag{1.3.4}$$

(i.e. those involving fewer than k normal derivatives ∂_t) on the initial hypersurface.

Theorem 1.3.3 *Assume that $\phi_0, \dots \phi_{k-1}$ are all real analytic functions in some neighbourhood of a point x_0 and that G is a real analytic function of its arguments in a neighbourhood of $(x_0, 0, \{\partial^\alpha \phi_j(x_0)\}_{|\alpha|+j\le k, j<k})$. Then there exists a unique real analytic function which satisfies* (1.3.3)-(1.3.2) *in some neighbourhood of the point x_0.*

Notice that the non-characteristic condition ensures that the k^{th} normal derivative $\partial_t^k u(x, 0)$ is determined by the data through the equation. Differentiation of (1.3.2) gives further relations which can be shown to determine all derivatives of the solution at $t = 0$, and the theorem can be proved by showing that the resulting Taylor series defines a real-analytic solution of the equation. Read section 1C of the book of Folland for the full proof.

In the case of first order equations with real coefficients the method of characteristics gives an alternative method of attack which does not require real analyticity. In this case we consider a pde of the form

$$\sum_{j=1}^{n} a_j(x,u)\partial_j u = b(x,u) \tag{1.3.5}$$

with data

$$u(x) = \phi(x)\,, \quad x \in \mathcal{S} \tag{1.3.6}$$

where $\mathcal{S} \subset \mathbb{R}^n$ is a hypersurface, given in paramteric form as $x_j = g_j(\sigma)\,, \sigma = (\sigma_1, \dots \sigma_{n-1}) \in \mathbb{R}^{n-1}$. (Think of $\mathcal{S} = \{x_n = 0\}$ parametrized by $g(\sigma_1, \dots \sigma_{n-1}) = (\sigma_1, \dots \sigma_{n-1}, 0)$.)

Theorem 1.3.4 *Let $\mathcal{S}$ be a C^1 hypersurface, and assume that the a_j, b, ϕ are all C^1 functions. Assume the non-characteristic condition:*

$$\Sigma_{j=1}^{n} a_j(x_0, \phi(x_0)) n_j(x_0) \neq 0$$

holds at a point $x_0 \in \mathcal{S}$. Then there is an open set $\mathcal{O}$ containing x_0 in which there exists a unique C^1 solution of (1.3.5) *which also satisfies* (1.3.6) *at all $x \in \mathcal{O} \cap \mathcal{S}$. If the non-characteristic condition holds at all points of $\mathcal{S}$, then there is a unique solution of* (1.3.5)-(1.3.6) *in an open neighbourhood of $\mathcal{S}$.*

This is proved by considering the characteristic curves which are obtained by integrating the system of $n+1$ characteristic ordinary differential equations (ode):

$$\frac{dx_j}{ds} = a_j(x,z)\,, \quad \frac{dz}{ds} = b(x,z) \tag{1.3.7}$$

with data $x_j(\sigma, 0) = g_j(\sigma), z(\sigma, 0) = \phi(g(\sigma))$; let $(X(\sigma,s), Z(\sigma,s)) \in \mathbb{R}^n \times \mathbb{R}$ be this solution. Now compute the Jacobian matrix of the mapping $(\sigma, s) \mapsto X(\sigma, s)$ at the point $(\sigma, 0)$: it is the $n \times n$ matrix whose columns are $\{\partial_{\sigma_j} g\}_{j=1}^{n-1}$ and the vector $a = (a_1, \dots a_n)$, evaluated at $x = g(\sigma), z = \phi(g(\sigma))$. The non-characteristic condition implies that this matrix is invertible (a linear bijection) and hence, via the inverse function theorem, that the "restricted flow map" which takes $(\sigma, s) \mapsto X(\sigma, s) = x$ is locally invertible, with inverse $\sigma_j = \Sigma_j(x), s = S(x)$ and this allows one to recover the solution as $u(x) = Z(\Sigma(x), S(x))$. This just means we have found a locally unique characteristic curve passing through x, and have then found $u(x)$ by tracing its value back along the curve to a point $g(\Sigma(x))$ on the initial hypersurface.

1.4 Some worked problems

1. Consider the two-dimensional domain

$$G := \{(x,y) \mid R_1^2 < x^2 + y^2 < R_2^2\},$$

where $0 < R_1 < R_2 < \infty$. Solve the Dirichlet boundary value problem for the Laplace equation

$$\Delta u = 0 \text{ in } G,$$
$$u = u_1(\varphi),\ r = R_1,$$
$$u = u_2(\varphi),\ r = R_2,$$

where (r, φ) are polar coordinates. Assume that u_1, u_2 are smooth 2π-periodic functions on the real line.

Discuss the convergence properties of the series so obtained.

[*Hint: Use separation of variables in polar coordinates ($u = R(r)\Phi(\varphi)$), with perodic boundary conditions for the function Φ of the angle variable. Use an ansatz of the form $R(r) = r^\alpha$ for the radial function.*]

Answer As the hint suggests, we use radial coordinates and transform the Laplacian. Using this, our PDE becomes

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\varphi\varphi} = 0.$$

Using separation of variables as the hint suggests yields

$$R''\Phi + \frac{1}{r}R'\Phi + \frac{1}{r^2}R\Phi'' = 0$$

We multiply this equation by $\frac{r^2}{R\Phi}$ and rearrange to obtain an equality between an expression which depends only on r and an expression which depends only on φ. This implies that the two equations are equal to a constant (denoted λ^2):

$$\frac{r^2R''(r) + rR'(r)}{R(r)} = \frac{-\Phi''(\varphi)}{\Phi} = \lambda^2$$

We require a 2π periodicity for each value of the radial coordinate, so we require that Φ be 2π periodic, and thus obtain:

$$\Phi(\varphi) = A\sin(\lambda\varphi) + B\cos(\lambda\varphi),$$

with positve constants A, B. Since we assume the solution to be 2π-periodic it follows that λ must be an integer, and w.l.o.g. λ is non-negative.

For the other equation we use the Ansatz $R(r) = r^\alpha$ and obtain

$$\alpha(\alpha-1)r^\alpha + \alpha r^\alpha - \lambda^2 r^\alpha = 0$$

and thus

$$\alpha = \pm\lambda.$$

Therefore

$$R(r) = Cr^\lambda + Dr^{-\lambda}, \quad \text{if } \lambda > 0$$

and by considering the case $\lambda = 0$ separately

$$R(r) = E + F\ln(r), \quad \text{if } \lambda = 0.$$

As the equation is linear, the most general solution is

$$u(r,\varphi) = \sum_{\lambda=1}^{\infty} (A_\lambda r^\lambda + B_\lambda r^{-\lambda}) \sin(\lambda\varphi) + (C_\lambda r^\lambda + D_\lambda r^{-\lambda}) \cos(\lambda\varphi) + E_0 + F_0 \ln(r)$$

To account for the boundary conditions, we expand u_1 and u_2 as Fourier series:

$$u_1(\varphi) = \frac{a_0}{2} + \sum_{\lambda=1}^{\infty} a_\lambda \sin(\lambda\varphi) + b_\lambda \cos(\lambda\varphi),$$
$$u_2(\varphi) = \frac{c_0}{2} + \sum_{\lambda=1}^{\infty} c_\lambda \sin(\lambda\varphi) + d_\lambda \cos(\lambda\varphi).$$

(This is possible by the assumptions on $u_{1,2}$.) To enforce

$$u(R_1,\varphi) = u_1(\varphi), \quad u(R_2,\varphi) = u_2(\varphi)$$

comparison of the coefficients leads to

$$A_\lambda R_1^\lambda + B_\lambda R_1^{-\lambda} = a_\lambda, \quad A_\lambda R_2^\lambda + B_\lambda R_2^{-\lambda} = c_\lambda,$$
$$C_\lambda R_1^\lambda + D_\lambda R_1^{-\lambda} = b_\lambda, \quad C_\lambda R_2^\lambda + D_\lambda R_2^{-\lambda} = d_\lambda$$

and

$$E_0 + F_0 \ln(R_1) = \frac{a_0}{2},$$
$$E_0 + F_0 \ln(R_2) = \frac{c_0}{2}.$$

The first two equations result in

$$A_\lambda = \frac{R_1^\lambda a_\lambda - R_2^\lambda c_\lambda}{R_1^{2\lambda} - R_2^{2\lambda}}, \quad B_\lambda = \frac{R_1^{-\lambda} a_\lambda - R_2^{-\lambda} c_\lambda}{R_1^{-2\lambda} - R_2^{-2\lambda}},$$

and

$$C_\lambda = \frac{R_1^\lambda b_\lambda - R_2^\lambda d_\lambda}{R_1^{2\lambda} - R_2^{2\lambda}}, \quad D_\lambda = \frac{R_1^{-\lambda} b_\lambda - R_2^{-\lambda} d_\lambda}{R_1^{-2\lambda} - R_2^{-2\lambda}}.$$

From the last equation we obtain

$$E_0 = \frac{a_0 \ln(R_2) - c_0 \ln(R_1)}{2(\ln(R_2) - \ln(R_1))}, \quad F_0 = \frac{c_0 - a_0}{2(\ln(R_2) - \ln(R_1))}.$$

Since the $u_{1,2}$ are smooth and periodic, their Fourier coeficients are rapidly decreasing, i.e.

$$\sup_{\lambda\in\mathbb{N}} \lambda^N (|a_\lambda| + |b_\lambda| + |c_\lambda| + |d_\lambda|) < \infty$$

for any positive N. Now let $\rho = R_1/R_2 = (R_2/R_1)^{-1} \in (0,1)$, then the above formulae can be written as

$$A_\lambda = R_2^{-\lambda} \frac{c_\lambda - \rho^\lambda a_\lambda}{1 - \rho^{2\lambda}}, \quad B_\lambda = R_1^\lambda \frac{a_\lambda - \rho^\lambda c_\lambda}{1 - \rho^{2\lambda}},$$

and

$$C_\lambda = R_2^{-\lambda}\frac{d_\lambda - \rho^\lambda b_\lambda}{1-\rho^{2\lambda}}\,, \quad D_\lambda = R_1^\lambda \frac{b_\lambda - \rho^\lambda d_\lambda}{1-\rho^{2\lambda}}\,.$$

Since $\rho \in (0,1)$ it follows from these formulae that $|A_\lambda| \le R_2^{-\lambda}(|a_\lambda| + |c_\lambda|)/(1-\rho)$ and $|B_\lambda| \le R_1^\lambda(|a_\lambda| + |c_\lambda|)/(1-\rho)$ while $|C_\lambda| \le R_2^{-\lambda}(|b_\lambda| + |d_\lambda|)/(1-\rho)$ and $|D_\lambda| \le R_1^\lambda(|b_\lambda| + |d_\lambda|)/(1-\rho)$. As a consequence

$$\sup_{\lambda\in\mathbb{N}} \sup_{R_1\le r\le R_2} \lambda^N(r^\lambda|A_\lambda| + r^{-\lambda}|B_\lambda| + r^\lambda|C_\lambda| + r^{-\lambda}|D_\lambda|) < \infty$$

for any positive N. Therefore the series

$$\begin{aligned} u(r,\varphi) = &\sum_{\lambda=1}^{\infty}(A_\lambda r^\lambda + B_\lambda r^{-\lambda})\sin(\lambda\varphi) + (C_\lambda r^\lambda + D_\lambda r^{-\lambda})\cos(\lambda\varphi) \\ &+ E_0 + F_0\ln(r) \end{aligned}$$

converges absolutely and uniformly in the *closed* annulus $\overline{G}$, to define a continuous function $u \in C(\overline{G})$, which agrees with the given data on the boundary ∂G. Furthermore u is smooth in the open annulus G where it solves $\Delta u = 0$.

As a final comment on the method of solution, an alternative to separation of variables is to say that any *smooth* function $u(r,\varphi)$ *which is* 2π *periodic* in φ can be decomposed as

$$u(r,\varphi) = u_0(r) + \sum_{\lambda=1}^{\infty}\alpha_\lambda(r)\sin(\lambda\varphi) + \beta_\lambda(r)\cos(\lambda\varphi)\,,$$

with $\alpha_\lambda, \beta_\lambda$ rapidly decreasing so that term by term differentiation is allowed. Then substitute this into the equation to obtain equations for $u_0(r), \alpha_\lambda(r), \beta_\lambda(r)$ and the same answer will follow.

2. (i) State the local existence theorem for real-valued solutions of the first order quasi-linear partial differential equation

$$\sum_{j=1}^{n} a_j(x,u)\frac{\partial u}{\partial x_j} = b(x,u) \tag{1.4.1}$$

with data specified on a hypersurface S, including a definition of "non-characteristic" in your answer. Also define the characteristic curves for (1.4.1) and briefly explain their use in obtaining the solution.

(ii) For the linear constant coefficient case (i.e. all the functions $a_1,\ldots,a_n$, are real constants and $b(x,u) = cu + d$ for some real numbers c, d) and with the hypersurface S taken to be the hyperplane $x\cdot\nu = 0$ explain carefully the relevance of the non-characteristic condition to obtaining a solution via the method of characteristics.

(iii) Solve the equation

$$\frac{\partial u}{\partial y} + u\frac{\partial u}{\partial x} = 0,$$

with initial data $u(0,y) = -y$ prescribed on $x = 0$, for a real valued function. Describe the domain on which your solution is C^1 and comment on this in relation to the theorem stated in (i).

Answer (i)

Theorem 1.4.1 *Let $\mathcal{S}$ be a C^1 hypersurface, and assume that the a_j, b, ϕ are all C^1 functions. Assume the non-characteristic condition:*

$$\Sigma_{j=1}^n a_j(x_0, \phi(x_0))n_j(x_0) \neq 0$$

holds at a point $x_0 \in \mathcal{S}$. Then there is an open set $\mathcal{O}$ containing x_0 in which there exists a unique C^1 solution of (1.4.1) *which also satisfies*

$$u(x) = \phi(x)\,, \quad x \in \mathcal{S} \cap \mathcal{O}\,. \tag{1.4.2}$$

If the non-characteristic condition holds at all points of $\mathcal{S}$, then there is a unique solution of (1.4.1)-(1.3.6) *in an open neighbourhood of $\mathcal{S}$.*

The characteristic curves are obtained as the x component of the integral curves of the characteristic ode:

$$\frac{dx_j}{ds} = a_j(x,z)\,, \quad \frac{dz}{ds} = b(x,z) \tag{1.4.3}$$

with data $x_j(\sigma, 0) = g_j(\sigma), z(\sigma, 0) = \phi(g(\sigma))$; let $(X(\sigma, s), Z(\sigma, s)) \in \mathbb{R}^n \times \mathbb{R}$ be this solution. The characteristic curves starting at $g(\sigma)$ are the curves $s \mapsto X(\sigma, s)$. They are useful because the non-characteristic condition implies (via the inverse function theorem) that the "restricted flow map" which takes $(\sigma, s) \mapsto X(\sigma, s) = x$ is locally invertible, with inverse $\sigma_j = \Sigma_j(x), s = S(x)$ and this allows one to obtain the solution by tracing along the characteristic curve using the z component of the characteristic ode above. This gives the final formula: $u(x) = Z(\Sigma(x), S(x))$.

(ii) In the linear constant coefficient case the non-characteristic condition reads $a \cdot \nu \neq 0$, and the characteristic curves are lines with tangent vector $a = (a_1, \dots a_n)$, obtained by integrating the characteristic ode:

$$\frac{dx_j}{ds} = a_j\,, \quad \frac{dz}{ds} = b(x,z) = cz + d\,, \tag{1.4.4}$$

and taking the "x component". The flow map is the smooth function $\mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ given by

$$\Phi(s, x) = x + sa,$$

i.e. the solution of the characteristic ode starting at x. Parametrize the initial hyperplane as $x = \sum_{j=1}^{n-1} \sigma_j \gamma_j$, where the $\gamma_j \in \mathbb{R}^n$ are a linearly independent set of vectors in the plane (i.e. satisfying $\gamma_j \cdot \nu = 0$). The restricted flow map is just the restriction of the flow map to the initial hypersurface, i.e.

$$X(s, \sigma) = \sum_{j=1}^{n-1} \sigma_j \gamma_j + sa = J \begin{pmatrix} \sigma_1 \\ \vdots \\ \sigma_{n-1} \\ s \end{pmatrix}.$$

Notice that here J, the Jacobian of the linear mapping $(s, \sigma) \mapsto X(s, \sigma)$, is precisely the *constant* $n \times n$ matrix whose columns are $\{\gamma_1, \gamma_2, \dots \gamma_{n-1}, a\}$. But the non-characteristic condition $a \cdot \nu \neq 0$ is equivalent to invertibility of this matrix and consequently $X(s, \sigma_1, \dots, \sigma_{n-1}) = x$ is uniquely solvable for

$$s = S(x), \sigma_1 = \Sigma_1(x), \dots, \sigma_{n-1} = \Sigma_{n-1}(x)$$

as functions of x (i.e. X is a linear bijection). This means that given any point $x \in \mathbb{R}^n$ there is a unique characteristic passing through it which intersects the initial hyperplane at exactly one point. This detemines the solution u uniquely at x since by the chain rule $z(s, \sigma_1, \ldots, \sigma_{n-1}) = u(X(s, \sigma_1, \ldots, \sigma_{n-1}))$ satisfies

$$\frac{dz}{ds} = cz + d,$$

so the evolution of u along the characteristic curves is known.

(iii) The characteristic ode are

$$\frac{dx}{ds} = -z\,, \ \frac{dy}{ds} = 1\,, \ \frac{dz}{ds} = 0\,.$$

The initial hypersurface can be parametrized as $(x(\sigma), y(\sigma) = (0, \sigma)$ and the solutions of the characteristic ode with initial data $z(0, \sigma) = -\sigma$ are $x(s, \sigma) = -s\sigma$, $y(s, \sigma) = \sigma + s$ and $z(s, \sigma) = z(0, \sigma) = -\sigma$. The restricted flow map is therefore $X(s, \sigma) = (-s\sigma, s + \sigma)$. Inverting this leads to a quadratic and the solution is given explicitly as:

$$u(x, y) = -\frac{1}{2}y - \frac{1}{2}\sqrt{(y^2 + 4x)} \quad y > 0,$$

$$u(x, y) = -\frac{1}{2}y + \frac{1}{2}\sqrt{(y^2 + 4x)} \quad y < 0,$$

where $\sqrt{a}$ means positive square root of a. Both of these formulae define C^1 (even smooth) functions in the region $\{y^2 + 4x > 0\}$, and can be verified to solve $u_y + uu_x = 0$ there. The region $\{y^2 + 4x > 0\}$ includes open neighbourhoods of every point on the initial hypersurface $x = 0$ except for the point $x = 0 = y$: this fits in with the statement of the theorem since it is at this point, and only this point, that the non-characterisitic condition fails to hold. To solve the Cauchy problem it is necessary to match the initial data: *notice that the signs of the square roots in the solution given above are chosen to ensure that the initial data are taken on correctly.* It is necessary to choose one of the "branches", depending upon how the initial hypersurface $\{x = 0\}$ is approached. This means the solution is no longer globally smooth - it is discontinuous along the half line $\{x > 0, y = 0\}$. As in complex analysis this line of discontinuity (like a "branch-cut") could be chosen differently, e.g. the half line $\{y = x, x > 0\}$.

1.5 Example sheet 1

1. Write out the multinomial expansion for $(x_1 + \ldots x_n)^N$ and the n-dimensional Taylor expansion using multi-index notation.
2. Consider the problem of solving the heat equation $u_t = \Delta u$ with data $u(x, 0) = f(x)$. Is the non-characteristic condition satisfied? How about for the wave equation $u_{tt} = \Delta u$ with data $u(x, 0) = f(x)$ and $u_t(x, 0) = g(x)$? For which of these problems, and for which data, does the Cauchy-Kovalevskaya theorem ensure the existence of a local solution? How about the Cauchy problem for the Schrödinger equation?
3. (a) Find the characteristic vectors for the operator $P = \partial_1\partial_2$ $(n = 2)$. Is it elliptic? Do the same for $P = \sum_{j=1}^m \partial_j^2 - \sum_{j=m+1}^n \partial_j^2$ $(1 < m < n)$.
   (b) Let $\Delta = \sum_{j=1}^{n-1} \partial_j^2$ be the laplacian. For which vectors $a \in \mathbb{R}^{n-1}$ is the operator $P = \partial_t^2 u + \partial_t \sum_{j=1}^{n-1} a_j \partial_j u - \Delta u$ hyperbolic?

4. Solve the linear PDE $x_1 u_{x_2} - x_2 u_{x_1} = u$ with boundary condition $u(x_1, 0) = f(x_1)$ for f a C^1 function. Where is your solution valid? Classify the f for which a global C^1 solution exists. (Global solution here means a solution which is C^1 on all of $\mathbb{R}^2$.Solve the linear PDE $x_1 u_{x_2} - x_2 u_{x_1} = u$ with boundary condition $u(x_1, 0) = f(x_1)$ for f a C^1 function. Where is your solution valid? Classify the f for which a global C^1 solution exists. (Global solution here means a solution which is C^1 on all of $\mathbb{R}^2$.
5. Solve Cauchy problem for the semi-linear PDE $u_{x_1} + u_{x_2} = u^4$, $u(x_1, 0) = f(x_1)$ for f a C^1 function. Where is your solution C^1?
6. For the quasi-linear Cauchy problem $u_{x_2} = x_1 u u_{x_1}$, $u(x_1, 0) = x_1$
(a) Verify that the Cauchy-Kovalevskaya theorem implies existence of an analytic solution in a neighbourhood of all points of the initial hypersurface $x_2 = 0$ in $\mathbb{R}^2$,
(b) Solve the characteristic ODE and discuss invertibility of the restricted flow map $X(s,t)$ (this may not be possible explicitly),
(c) give the solution to the Cauchy problem (implicitly).
7. For the quasi-linear Cauchy problem $Au_{x_1} - (B - x_1 - u)u_{x_2} + A = 0$, $u(x_1, 0) = 0$:
(a) Find all points on the intial hypersurface where the Cauchy-Kovalevskaya theorem can be applied to obtain a local solution defined in a neighbourhood of the point.
(b) Solve the characteristic ODE and invert (where possible) the restricted flow map, relating your answer to (a).
(c) Give the solution to the Cauchy problem, paying attention to any sign ambiguities that arise.
(In this problem take A, B to be positive real numbers).
8. For the Cauchy problem

$$u_{x_1} + 4u_{x_2} = \alpha u \qquad u(x_1, 0) = f(x_1), \tag{1.5.1}$$

with C^1 initial data f, obtain the solution $u(x_1, x_2) = e^{\alpha x_2/4} f(x_1 - x_2/4)$ by the method of characteristics. For fixed x_2 write $u(x_2)$ for the function $x_1 \mapsto u(x_1, x_2)$ i.e. the solution restricted to "time" x_2. Derive the following *well-posedness* properties for solutions $u(x_1, x_2)$ and $v(x_1, x_2)$ corresponding to data $u(x_1, 0)$ and $v(x_1, 0)$ respectively:
(a) for $\alpha = 0$ there is *global well-posedness* in the supremum (or L^∞) norm *uniformly in time* in the sense that if for fixed x_2 the distance between u and v is taken to be

$$\|u(x_2) - v(x_2)\|_{L^\infty} \equiv \sup_{x_1} |u(x_1, x_2) - v(x_1, x_2)|$$

then

$$\|u(x_2) - v(x_2)\|_{L^\infty} \leq \|u(0) - v(0)\|_{L^\infty} \qquad \text{for all } x_2.$$

Is the inequality ever strict?
(b) for all α there is *well-posedness* in supremum norm *on any finite time interval* in the sense that for any time interval $|x_2| \leq T$ there exists a number $c = c(T)$ such that

$$\|u(x_2) - v(x_2)\|_{L^\infty} \leq c(T)\|u(0) - v(0)\|_{L^\infty}.$$

and find $c(T)$. Also, for different α, when can c be assumed independent of time for positive (respectively negative) times x_2?
(c) Try to do the same for the L^2 *norm*, i.e. the norm defined by

$$\|u(x_2) - v(x_2)\|^2_{L^2(dx_1)} = \int |u(x_1, x_2) - v(x_1, x_2)|^2 dx_1.$$

9. For which real numbers a can you solve the Cauchy problem

$$u_{x_1} + u_{x_2} = 0 \qquad u(x_1, ax_1) = f(x_1)$$

for any C^1 function f. Explain both in terms of the non-characteristic condition and by explicitly trying to invert the (restricted) flow map, interpreting your answer in relation to the line $x_2 = ax_1$ on which the initial data are given.

10. (a). Consider the equation

$$u_{x_1} + nu_{x_2} = f \tag{1.5.2}$$

where n is an integer and f is a smooth function which is 2π- periodic in both variables:

$$f(x_1 + 2\pi, x_2) = f(x_1, x_2 + 2\pi) = f(x_1, x_2).$$

Apply the method of characteristics to find out for which f there is a solution which is also 2π- periodic in both variables:

$$u(x_1 + 2\pi, x_2) = u(x_1, x_2 + 2\pi) = u(x_1, x_2).$$

(b) Consider the problem in part (a) using fourier series representations of f and u (both 2π- periodic in both variables) and compare your results.
(c)* What can you say about the case when n is replaced by an *irrational* number ω? [Hint: look in *http : //en.wikipedia.org/wiki/Diophantine_approximation* for the definition of Liouville number, and use this as a condition to impose on ω and investigate the consequences for solving (1.5.2).]

2 Background analysis

2.1 Fourier series

Consider the following spaces of 2π-periodic functions on the real line:

$$C^r_{per}([-\pi, \pi]) = \{u \in C^r(\mathbb{R}) : u(x + 2\pi) = u(x)\} ,$$

for $r \in [0, \infty]$. The case $r = 0$ is the continuous 2π-periodic functions, while the case $r = \infty$ is ths smooth 2π-periodic functions. For functions $u = u(x_1, \ldots x_n)$ we define the corresponding spaces $C^r_{per}([-\pi, \pi]^n)$ of C^r functions which are 2π-periodic in each coordinate. (All of these definitions generalize in obvious ways for classes of functions with periods other than 2π, e.g. $C^r_{per}(\prod_{j=1}^n [0, L_j])$ consists of C^r functions $u = u(x_1, \ldots x_n)$ which are L_j-periodic in x_j.)

Given a function $u \in C^\infty_{per}([-\pi, \pi])$ the Fourier coefficients are the sequence of numbers $\hat{u}_m = \hat{u}(m)$ given by

$$\hat{u}(m) = \hat{u}_m = \frac{1}{2\pi} \int_{-\pi}^{+\pi} e^{-imx} u(x)\, dx\,, \qquad m \in \mathbb{Z}\,.$$

Integration by parts gives the formula $\widehat{\partial^\alpha u}(m) = (im)^\alpha \hat{u}(m)$ for positive integral α, which shows that the sequence of Fourier coefficients is a rapidly decreasing (bi-infinite) sequence: this means that $\hat{u} \in s(\mathbb{Z})$ where

$$s(\mathbb{Z}) = \{\hat{u} : \mathbb{Z} \to \mathbb{C} \text{ such that } |\hat{u}|_\alpha = \sup_{m\in\mathbb{Z}} |m^\alpha \hat{u}(m)| < \infty \ \forall \alpha \in \mathbb{Z}_+\}\,.$$

This in turn means that the series $\sum_{m\in\mathbb{Z}} \hat{u}(m)e^{imx}$ converges absolutely and uniformly to a smooth function. The central fact about Fourier series is that this series actually converges to u, so that each $u \in C^\infty_{per}([-\pi,\pi])$ can be represented as:

$$u(x) = \sum \hat{u}(m)e^{imx}\,, \quad \text{where} \quad \hat{u}(m) = \frac{1}{2\pi}\int_{-\pi}^{+\pi} e^{-imx}u(x)\,dx\,.$$

The whole development works for periodic functions $u = u(x_1, \dots x_n)$ with the sequence space generalized to

$$s(\mathbb{Z}^n) = \{\hat{u} : \mathbb{Z}^n \to \mathbb{C} \text{ such that } |\hat{u}|_\alpha = \sup_{m\in\mathbb{Z}^n} |m^\alpha \hat{u}(m)| < \infty \ \forall \alpha \in \mathbb{Z}^n_+\}\,.$$

Here we use multi-index notation, in terms of which we have:

Theorem 2.1.1 *The mappping*

$$\begin{aligned} C^\infty_{per}([-\pi,\pi]^n) &\to s(\mathbb{Z}^n)\,, \\ u &\mapsto \hat{u} = \{\hat{u}(m)\}_{m\in\mathbb{Z}^n} \qquad \textit{where } \hat{u}(m) = \frac{1}{(2\pi)^n}\int_{[-\pi,\pi]^n} e^{-im\cdot x}u(x)\,dx \end{aligned}$$

is a linear bijection whose inverse is the map which takes $\hat{u}$ *to* $\sum_{m\in\mathbb{Z}^n} \hat{u}(m)e^{im\cdot x}$ *and the following hold:*

1. $u(x) = \sum_{m\in\mathbb{Z}^n} \hat{u}(m)e^{im\cdot x}$ *where* $\hat{u}(m) = \frac{1}{(2\pi)^n}\int_{[-\pi,\pi]^n} e^{-im\cdot x}u(x)\,dx$ *(Fourier inversion),*

2. $\widehat{\partial^\alpha u}(m) = (im)^\alpha \hat{u}(m)$ *for all* $m \in \mathbb{Z}^n, \alpha \in \mathbb{Z}^n_+$*,*

3. $\frac{1}{(2\pi)^n}\int_{[-\pi,\pi]^n} |u(x)|^2\,dx = \sum_{m\in\mathbb{Z}^n} |\hat{u}(m)|^2$ *(Parseval-Plancherel).*

2.2 Fourier transform

Define the Schwartz space of test functions:

$$\mathcal{S}(\mathbb{R}^n) = \{u \in C^\infty(\mathbb{R}^n) : |u|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta u(x)| < \infty\,,\ \forall \alpha \in \mathbb{Z}^n_+, \beta \in \mathbb{Z}^n_+\,.\}$$

This is a convenient space on which to define the Fourier transform because of the fact that Fourier integrals interchange rapidity of decrease with smoothness, so the space of functions which are smooth and rapidly decreasing is invariant under Fourier transform:

Theorem 2.2.1 *The Fourier transform, i.e. the mapping*

$$\begin{aligned} \mathcal{F} : \mathcal{S}(\mathbb{R}^n) &\rightarrow \mathcal{S}(\mathbb{R}^n)\,, \\ u &\mapsto \hat{u} \qquad \text{where } \hat{u}(\xi) = \mathcal{F}_{x\to\xi}(u(x)) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx \end{aligned}$$

is a linear bijection whose inverse is the map $\mathcal{F}^{-1}$ which takes v to the function $\check{v} = \mathcal{F}^{-1}(v)$ given by

$$\check{v}(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} e^{+i\xi\cdot x} v(\xi)\, d\xi\,,$$

and the following hold:

1. $u(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \hat{u}(\xi) e^{i\xi\cdot x} d\xi$ *where* $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x} u(x)\, dx$ *(Fourier inversion),*

2. $\widehat{\partial^\alpha u}(\xi) = \mathcal{F}_{x\to\xi}(\partial^\alpha u(x)) = (i\xi)^\alpha \hat{u}(\xi)$ *and* $(\partial^\alpha \hat{u})(\xi) = \mathcal{F}_{x\to\xi}((-ix)^\alpha u(x))$ *for all* $x, \xi \in \mathbb{R}^n, \alpha \in \mathbb{Z}^n_+$*,*

3. $\int_{\mathbb{R}^n} \hat{v}(\xi) u(\xi)\, d\xi = \int_{\mathbb{R}^n} v(x) \hat{u}(x)\, dx\,,$

4. $\frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \overline{\hat{v}(\xi)} \hat{u}(\xi)\, d\xi = \int_{\mathbb{R}^n} \overline{v(x)} u(x)\, dx$ *(Parseval-Plancherel),*

5. $\widehat{u * v} = \hat{u}\hat{v}$ *where* $u * v = \int u(x-y) v(y)\, dy$ *(convolution).*

2.3 Banach spaces

A norm on a vector space X is a real function $x \mapsto \|x\|$ such that

1. $\|x\| \geq 0$ with equality iff $x = 0$,
2. $\|cx\| = |c|\|x\|$ for all $c \in \mathbb{C}$,
3. $\|x + y\| \leq \|x\| + \|y\|\,.$

(If the first condition is replaced by the weaker requirement $1'$ that $\|x\| \geq 0$ then the modified conditions $1', 2, 3$ define a semi-norm.) A normed vector space is a metric space with metric $d(x,y) = \|x - y\|$. Recall that a metric on X is a map $d : X \times X \rightarrow [0, \infty)$ such that

1. $d(x,y) \geq 0$ with equality iff $x = y$,
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq d(x,y) + d(y,z)$ for all x, y, z in X.

(This definition does not require that X be a vector space.) The metric space (X, d) is complete if every Cauchy sequence has a limit point: to be precise if $\{x_j\}_{j=1}^{\infty}$ has the property that $\forall \epsilon > 0$ there exists $N(\epsilon) \in \mathbb{N}$ such that $j, k \geq N(\epsilon) \implies d(x_j, x_k) < \epsilon$ then there exists $x \in X$ such that $\lim_{j\to\infty} d(x_j, x) = 0$.

Definition 2.3.1 *A Banach space is a normed vector space which is complete (using the metric $d(x, y) = \|x - y\|$).*

Examples are

- $\mathbb{C}^n$ with the Euclidean norm $\|z\| = (\sum_j |z_j|^2)^{\frac{1}{2}}$.
- $C([a, b])$ with $\|f\| = \sup_{[a,b]} |f(x)|$ (uniform norm).
- Spaces of p-summable (bi-infinite) sequences $\{u_m = u(m)\}_{m\in\mathbb{Z}}$

  $$l^p(\mathbb{Z}) = \{u : \mathbb{Z} \to \mathbb{C} \text{ such that } \|u\|_p = (\sum |u(m)|^p)^{\frac{1}{p}} < \infty\}$$

  and generalizations such as $l^p(\mathbb{Z}^n)$ and $l^p(\mathbb{N})$.
- Spaces of measurable L^p functions for $1 \leq p < \infty$

  $$L^p(\mathbb{R}^n) = \{u : \mathbb{R}^n \to \mathbb{C} \text{ measurable with } \|u\|_p = (\int |u(x)|^p \, dx)^{\frac{1}{p}} < \infty\}$$

  and generalizations such as $L^p([-\pi, \pi]^n)$ and $L^p([0, \infty))$ etc. For $p = \infty$ the space $L^\infty(\mathbb{R}^n)$ consists of measurable functions which are bounded on the complement of a null set, and the least such bound is called the essential supremum and gives the norm $\|u\|_{L^\infty}$. In this example we identify functions which agree on the complement of a null set (almost everywhere). Read the appendix for an informal introduction to the Lebesgue spaces and a list of results from integration that we will use in this course[1].

Completeness is important because methods for proving that an equation has a solution typically produce a sequence of "approximate solutions", e.g. Picard iterates for the case of ode. If this sequence can be shown to be Cauchy in some norm then completeness ensures the existence of a limit point which is the putative solution, and in good situations can be proved to be a solution. The basic result is the *contraction mapping principle* which can be used to prove existence of solutions of equations of the form $Tx = x$, i.e. to find fixed points of mappings $T : X \to X$ defined on complete (non-empty) metric spaces:

Theorem 2.3.2 *Let (X, d) be a complete, non-empty, metric space and $T : X \to X$ a map such that*

$$d(Ty_1, Ty_2) \leq k d(y_1, y_2)$$

with $k \in (0, 1)$. Then T has a unique fixed point in X; in fact if $y_0 \in X$, then $T^m y_0$ converges to a fixed point as $m \to \infty$.

[1] You will not be examined on any subtle points connected with the Lebesgue integral

Proof We first prove uniqueness of any fixed point: notice that if $Ty_1 = y_1$ and $Ty_2 = y_2$ then the contraction property implies

$$d(y_1, y_2) = d(Ty_1, Ty_2) \leq kd(y_1, y_2)$$

and therefore, since $0 < k < 1$, we have $d(y_1, y_2) = 0$ and so $y_1 = y_2$.

To prove existence, firstly, by the triangle inequality:

$$\begin{aligned} d(T^{n+r+1}y_0, T^n y_0) &\leq \sum_{m=0}^{r} d(T^{n+m+1}y_0, T^{n+m}y_0) \\ &\leq \sum_{m=0}^{r} k^m d(T^{n+1}y_0, T^n y_0) \leq \sum_{m=0}^{r} k^{n+m} d(Ty_0, y_0)\,. \end{aligned}$$

But $0 < k < 1$ implies that $\sum_{m\geq 0} k^m = \frac{1}{1-k} < \infty$ and hence $T^m y_0$ forms a Cauchy sequence in X. So by completeness of X, $T^m y_0 \to y$ some y. But then $T^{m+1}y_0 \to Ty$, because T is continuous by the contraction property, and so $Ty = y$ and y is a fixed point. Thus T has a unique fixed point. □

Now review the proof of the existence theorem for ode via the contraction mapping theorem in the Banach space of continuous functions with the uniform norm in §2.8. Theorem 2.8.4 says that for ode defined by Lipschitz vector fields the solutions vary continuously with the initial data: this is the crucial property of *well-posedness.* For pde the issue is more subtle as will now be discussed.

Norms are used in the definition of well posed: if a pde can be solved for a solution u which is uniquely determined by some set of initial and/or boundary data $\{f_j\}$ then the problem is said to be well posed in a norm $\|\cdot\|$ if the solution changes a small amount in this norm as the data change. This would be satisfied if for example for any other solution v determined by data $\{g_j\}$ there holds:

$$\|u - v\| \leq C\Big(\sum_j \|f_j - g_j\|_j\Big)\,, \qquad \text{for some } C > 0\,, \tag{2.3.1}$$

where $\|\cdot\|_j$ are some collection of norms which measure what kind of changes of data produce small changes of the solution. *Finding the appropriate norms such that* (2.3.1) *holds for a given problem is a crucial part of understanding the problem - they are generally not known in advance.* Once this is understood it is helpful with development of numerical methods for solving problems on computers, and tells you in an experimental situation how accurately you need to measure the data to make a good prediction.

To fix ideas consider the problem of solving an evolution equation of the form

$$\partial_t u = P(\partial_x)u$$

where P is a constant coefficient polynomial; e.g. the case $P(\partial_x) = i\partial_x^2$ corresponds to the Schrödinger equation $\partial_t u = i\partial_x^2 u$. If we are solving this with periodic boundary conditions $u(x,t) = u(x+2\pi,t)$ and with given initial data $u(x,0) = u_0(x)$ for $u_0 \in C^\infty_{per}([-\pi,\pi])$. *Formally* the solution can be given as

$$u(x,t) = \sum_{m\in\mathbb{Z}} e^{tP(im)+imx}\hat{u_0}(m) \tag{2.3.2}$$

and if the initial data $u_o = \sum \hat{u_0}(m)e^{imx}$ is a finite sum of exponentials then (2.3.2) is easily seen to define a solution since it reduces also to a finite sum. In the general case it is necessary to investigate convergence of the sum so that it does define a solution, then to prove uniqueness of this solution, and finally to find norms for which well-posedness holds. For this final step the Parseval identity is often very helpful, and for the case of the Schrödinger equation the series (2.3.2) does indeed define a solution for smooth periodic data u_0, v_0 and

$$\max_{t\in\mathbb{R}} \int |u(x,t)-v(x,t)|^2\,dx \le \int |u_0(x)-v_0(x)|^2\,dx\,.$$

This inequality would be interpreted as saying that the Schrödinger equation is well posed in L^2 (globally in time since there is no restriction on t.) In question 7 of sheet II you are asked to prove that the solutions are unique.

In general an equation defines a well posed problem with respect to specific norms, which encode certain aspects of the behaviour of the solutions and have to be found as part of the investigation: *the property of being well posed depends on the norm.* This is related to the fact that norms on infinite dimensional vector spaces (like spaces of functions) can be inequivalent (i.e. can correspond to different notions of convergence), unlike in Euclidean space $\mathbb{R}^n$.

2.4 Hilbert spaces

A Hilbert space is a Banach space which is also an inner product space: the norm arises as $\|x\| = (x,x)^{\frac{1}{2}}$ where $(\cdot,\cdot) : X \times X \to \mathbb{C}$ satisfies:

1. $(x,x) \ge 0$ with equality iff $x = 0$,
2. $\overline{(x,y)} = (y,x)$,
3. $(ax+by,z) = \overline{a}\,(x,z) + \overline{b}\,(y,z)$ and $(x,ay+bz) = a\,(x,y) + b\,(x,z)$ for complex numbers a, b and vectors x, y, z.

(Funtions $X \times X \to \mathbb{C}$ like this which are linear in the second variable and anti-linear in the first are sometimes called sesqui-linear.) Crucial properties of the inner product in a Hilbert space are the Cauchy-Schwarz inequality $|\,(x,y)\,| \le \|x\|\|y\|$ and the fact that the inner product can be recovered from the norm via

$$(x,y) = \frac{1}{4}\left(\|x+y\|^2 - \|x-y\|^2 - i\|x+iy\|^2 + i\|x-iy\|^2\right)\,, \qquad \text{(polarization)}\,.$$

Examples include $l^2(\mathbb{Z}^n)$ with inner product $\sum_m \overline{u(m)}v(m)$ and $L^2(\mathbb{R}^n)$ with inner product $(u,v) = \int \overline{u(x)}v(x)\,dx$. Another example is the Sobolev spaces: firstly in the periodic case

$$H^1(\mathbb{R}/2\pi\mathbb{Z}) = \{u \in L^2([-\pi,\pi]) : \|u\|_1^2 = \sum_{m\in\mathbb{Z}} (1+|m|^2)|\hat{u}(m)|^2 < \infty\}, \quad (2.4.1)$$

where $u = \sum \hat{u}(m)e^{imx}$ is the Fourier representation, and secondly

$$H^1(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \|u\|_1^2 = \int_{\mathbb{R}^n} (1+|\xi|^2)|\hat{u}(\xi)|^2\,d\xi < \infty\}, \quad (2.4.2)$$

where $\hat{u}$ is the Fourier transform.

The new structure in Hilbert (as compared to Banach) spaces is the notion of orthogonality coming from the the inner product. A set of vectors $\{e_n\}$ is called orthonormal if $(e_n, e_m) = \delta_{nm}$. We will consider only Hilbert spaces which have a countable orthonormal basis $\{e_n\}$ (*separable* Hilbert spaces). In such spaces it is possible to decompose arbitrary elements as $u = \sum u_n e_n$ where $u_n = (e_n, u)$. (The case of Fourier series with $e_m(x) = e^{imx}/\sqrt{2\pi}, m \in \mathbb{Z}$ is an example.) The Parseval identity in abstract form reads $\|u\|^2 = \sum |(e_n,u)|^2$ and:

Theorem 2.4.1 *Given an orthonormal set* $\{e_n\}$ *the following are equivalent:*

- $(e_n, u) = 0\ \forall n$ *implies* $u = 0$, *(completeness)*
- $\|u\|^2 = \sum |(e_n,u)|^2\ \forall u \in X$, *(Parseval),*
- $u = \sum (e_n, u)\, e_n\ \forall u \in X$ *(orthonormal basis).*

A closed subspace $X_1 \subset X$ of a Hilbert space is also a Hilbert space, and there is an orthogonal decomposition

$$X = X_1 \oplus X_1^\perp$$

where $X_1^\perp = \{y \in X : (x_1, y) = 0 \forall x_1 \in X_1\}$. This means that any $x \in X$ can be written uniquely as $x = x_1 + y$ with $x_1 \in X_1$ and $y \in X_1^\perp$, and there is a correponding projection $P_{X_1}x = x_1$.

Associated to a Hilbert space X is its dual space X' which is defined to be the space a bounded linear maps:

$$X' = \{L : X \to \mathbb{C}\,,\ \text{with } L \text{ linear and } \|L\| = \sup_{x\in X, \|x\|=1} |Lx| < \infty\}.$$

The definition of the norm on X' ensures that $|L(x)| \le \|L\|\|x\|$.

Theorem 2.4.2 (Riesz representation) *Given a bounded linear map* L *on a Hilbert space* X *there exists a unique vector* $y \in X$ *such that* $Lx = (y,x)$*; also* $\|L\| = \|y\|$. *The correspondence between* L *and* y *gives an identification of the dual space* X' *with the original Hilbert space* X.

A generalization of this (for non-symmetric situations) is:

Theorem 2.4.3 (Lax-Milgram lemma) *Given a bounded linear map $L : X \to \mathbb{R}$ on a Hilbert space X, and a bilinear map $B : X \times X \to \mathbb{R}$ which satisfies (for some positive numbers $\|B\|, \gamma$):*

- $|B(x,y)| \leq \|B\|\|x\|\|y\| \quad \forall x, y \in X$ *(continuity),*
- $B(x,x) \geq \gamma \|x\|^2 \quad \forall x \in X$ *(coercivity),*

there exists a unique vector $y \in X$ such that $Lx = B(y,x) \forall x \in X$.

A bounded linear operator $B : X \to X$ means a linear map $X \to X$ with the property that there exists a number $\|B\| \geq 0$ such that $\|Bu\| \leq \|B\|\|u\| \, \forall u \in X$. As in Sturm-Liouville theory we say a bounded linear operator is diagonalizable if there is an orthonormal basis $\{e_n\}$ such that $Be_n = \lambda_n e_n$ for some collection of complex numbers λ_n which are the eigenvalues.

2.5 Distributions

Definition 2.5.1 *A periodic distribution $T \in C^\infty_{per}([-\pi, \pi]^n)'$ is a continuous linear map $T : C^\infty_{per}([-\pi, \pi]^n) \to \mathbb{C}$, where continuous means that if f_n and all its partial derivatives $\partial^\alpha f_n$ converge uniformly to f then $T(f_n) \to T(f)$. Here we call $C^\infty_{per}([-\pi, \pi]^n)$ the space of test functions.*

A tempered distribution $T \in \mathcal{S}'(\mathbb{R}^n)$ is a continuous linear map $T : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$, where continuous means that if $\|f_n - f\|_{\alpha,\beta} \to 0$ for every Schwartz semi-norm then $T(f_n) \to T(f)$. Here we call $\mathcal{S}(\mathbb{R}^n)$ the space of test functions.

In both cases for $x_0 \in \mathbb{R}^n$ any fixed point (which may be taken to lie in $[-\pi, \pi]^n$ in the periodic case) the Dirac distribution defined by $\delta_{x_0}(f) = f(x_0)$ gives an example.

Remark 2.5.2 *The notion of convergence on $C^\infty_{per}([-\pi, \pi]^n$ and $\mathcal{S}(\mathbb{R}^n)$ used in this definition makes these spaces into topological spaces in which the convergence must be with respect to a countable family of semi-norms. These are examples of Frechet spaces, a class of topological vector spaces which generalize the notion of Banach space by using a countable family of semi-norms rather than a single norm to define a notion of convergence. Using this notion of convergence one can check that the Fourier transform $\mathcal{F}$ is continuous as is its inverse, and the Fourier inversion theorem can be summarized by the assertion that $\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n)$ is a linear homeomorphism with inverse $\mathcal{F}^{-1}$.*

Remark 2.5.3 *Notice that integrable functions define distributions in a natural way: in the simplest case if g is continuous 2π-periodic function then the formula $T_g(f) = \int_{[-\pi,\pi]} g(x) f(x) \, dx$ defines a periodic distribution and clearly the mapping*

$g \mapsto T_g$ is an injection of $C_{per}([-\pi,\pi])$ into $(C^{\infty}_{per}([-\pi,\pi]))'$. Similarly if g is absolutely integrable on $\mathbb{R}^n$ then the formula $T_g(f) = \int_{\mathbb{R}^n} g(x)f(x)\,dx$ defines a tempered distribution. The mapping $g \mapsto T_g$ is, properly interpreted, injective: if $g \in L^1(\mathbb{R}^n)$ then $T_g(f) = 0$ for all $f \in \mathcal{S}(\mathbb{R}^n)$ implies that $g = 0$ almost everywhere. On account of this remark distributions are often called "generalized functions". The Dirac example indicates that there are distrubutions which do not arise as T_g.

Remark 2.5.4 *In these definitions distributions are elements of the dual space of a space of test functions with a specified notion of convergence (a topology). Another frequently used class of distributions is the dual space of $C_0^{\infty}(\mathbb{R}^n)$ the space of compactly supported smooth functions, topologized as follows: $f_n \to f$ in C_0^{∞} if there is a fixed compact set K such that all f_n, f are supported in K and if all partial derivatives of $\partial^{\alpha} f_n$ converge (uniformly) to $\partial^{\alpha} f$. This class of distributions is more convenient for some purposes, but not for using the Fourier transform, for which purpose the tempered distributions are most convenient because of remark 2.5.2, which allows the fourier transform to be defined on tempered distributions "by duality" as we now discuss.*

Operations are defined on distributions by using duality to transfer them to the test functions, e.g.:

- Given $T \in \mathcal{S}'(\mathbb{R}^n)$ an arbitrary partial derivative $\partial^{\alpha} T$ is defined by $\partial^{\alpha} T(f) = (-1)^{|\alpha|} T(\partial^{\alpha} f)$.
- Given $T \in \mathcal{S}'(\mathbb{R}^n)$ its fourier transform $\hat{T}$ is defined by $\hat{T}(f) = T(\hat{f})$.
- Given $T \in \mathcal{S}'(\mathbb{R}^n)$ and $\chi \in \mathcal{S}(\mathbb{R}^n)$ the distribution χT is defined by $\chi T(f) = T(\chi f)$. This is also the definition if χ is a polynomial - it makes sense because a polynomial times a Schwartz function is again a Schwartz function.

It is useful to check, with reference to the fact in remark 2.5.3 that distributions are generalized functions, that all such defintions of operations on distributions are designed to extend the corresponding definitions on functions: e.g. for a Schwartz function g we have

$$\partial^{\alpha} T_g = T_{\partial^{\alpha} g}\,,$$

where on the left ∂^{α} means distributional derivative while on the right it is the usual derivative from calculus applied to the test function g. The same principle is behind the other definitions.

There are various alternate notations used for distributions:

$$T(f) \;=\; \langle T\,,\, f\rangle \;=\; \int T(x)f(x)\,dx$$

where in the right hand version it should be remembered that the expression is purely formal in general: the putative function $T(x)$ has not been defined, and the integral notation is not an integral - just shorthand for the duality pairing of the definition. It is nevertheless helpful to use it to remember some formulae: for example the formula for the distributional derivative takes the form

$$\partial^\alpha T(f) \;=\; \int \partial^\alpha T(x) f(x)\, dx \;=\; (-1)^{|\alpha|} \int T(x) \partial^\alpha f(x)\, dx = (-1)^{|\alpha|} T(\partial^\alpha f)\,,$$

which is "familiar" from integration by parts. The formula $\int \delta(x - x_0) f(x)\, dx = f(x_0)$ and related ones are to be understood as formal expressions for the proper definition of the delta distribution above.

2.6 Positive distributions and Measures

In this section[2] we restrict to 2π-periodic distributions on the real line for simplicity. The delta distribution δ_{x_0} has the property that if $f \geq 0$ then $\delta_{x_0}(f) \geq 0$; such distributions are called positive. Positive distributions have an important continuity property as a result: if T is any positive periodic distribution, then since

$$-\|f\|_{L^\infty} \leq f(x) \leq \|f\|_{L^\infty}\,, \qquad \|f\| = \sup |f(x)|$$

for each $f \in C^\infty_{per}([-\pi, \pi])$ it follows from positivity that $T(\|f\|_{L^\infty} \pm f) \geq 0$ and hence by linearity that

$$-c\|f\|_{L^\infty} \leq T(f) \leq c\|f\|_{L^\infty}$$

where $c = T(1)$ is a positive number. This inequality, applied with f replaced by $f - f_n$, means that if $f_n \to f$ uniformly then $T(f_n) \to T(f)$, i.e. *positive distributions* are automatically continuous with respect to *uniform convergence*, in strong contrast to the continuity property required in the original definition. In fact this new continuity property ensures that a positive distribution can be extended uniquely as a map

$$L : C_{per}([-\pi, \pi]) \to \mathbb{R}$$

i.e. as a continuous linear functional on the space of continuous functions. This extension is an immediate consquence of the density of smooth functions in the continuous functions in the uniform norm (which can be deduced from the Weierstrass approximation theorem). A much more lengthy argument allows such a functional to be extended as an integral $L(f) = \int f\, d\mu$ which is defined for a class of measurable functions f which contains and is bigger than the class of continuous functions. To conclude: positive distributions automatically extend to define continuous linear functional on the space of continuous functions, and hence can be identified with a class of *measures* (Radon measures) which can be used to integrate much larger classes of functions (extending further the domain of the original distribution).

[2]This is an optional section, for background only

2.7 Sobolev spaces

We define the Sobolev spaces for $s = 0, 1, 2, \ldots$ on various domains:
On $\mathbb{R}^n$*:* we have the following equivalent definitions:

$$\begin{aligned} H^s(\mathbb{R}^n) &= \{u \in L^2(\mathbb{R}^n) : \|u\|^2_{H^s} = \sum_{\alpha:|\alpha|\leq s} \|\partial^\alpha u\|^2_{L^2} < \infty\} \\ &= \{u \in L^2(\mathbb{R}^n) : \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^s |\hat{u}(\xi)|^2 \, d\xi < \infty\} \\ &= \overline{C_0^\infty(\mathbb{R}^n)}^{\|\cdot\|_{H^s}} . \end{aligned}$$

In the first line the partial derivatives are taken in the distributional sense: the precise meaning is that all *distributional (=weak)* partial derivatives up to order s of the distribution T_u determined by u are distributions which are determined by square integrable funtions which are designated $\partial^\alpha u$ (i.e. $\partial^\alpha T_u = T_{\partial^\alpha u}$ with $\partial^\alpha u \in L^2$ in the notation introduced previously). The final line means that H^s is the closure of the space of smooth compactly supported functions $C_0^\infty(\mathbb{R}^n)$ in the Sobolev norm $\|\cdot\|_{H^s}$. The quantity $\tilde{\|}u\tilde{\|}^2_{H^s} = \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^s |\hat{u}(\xi)|^2 \, d\xi$ appearing in the middle definition defines a norm which is equivalent to the norm $\|u\|_{H^s}$ appearing in the first definition. (Recall that $\|\cdot\|$ and $\tilde{\|}\cdot\tilde{\|}$ are equivalent if there exist positive numbers C_1, C_2 such that $\|u\| \leq C_1 \tilde{\|}u\tilde{\|}$ and $\tilde{\|}u\tilde{\|} \leq C_2 \|u\|$ for all vectors u; equivalent norms give rise to identical notions of convergence (i.e. they define the same topologies).

Theorem 2.7.1 *For $s = 0, 1, 2 \ldots$ the Sobolev space $H^s(\mathbb{R}^n)$ is a Hilbert space, and so complete in either of the norms*

$$\|u\|^2_{H^s} = \sum_{\alpha:|\alpha|\leq s} \|\partial^\alpha u\|^2_{L^2} \quad \text{or} \quad \tilde{\|}u\tilde{\|}^2_{H^s} = \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^s |\hat{u}(\xi)|^2 \, d\xi$$

which are equivalent. Given any $u \in H^s(\mathbb{R}^n)$ there exists a sequence u_ν of $C_0^\infty(\mathbb{R}^n)$ functions such that $\|u - u_\nu\|_{H^s} \to 0$ as $\nu \to +\infty$. If $u \in H^s(\mathbb{R}^n)$ for $s > \frac{n}{2} + k$ with $k \in \mathbb{Z}_+$ then $u \in C^k(\mathbb{R}^n)$ and there exists $C > 0$ such that

$$\|u\|_{C^k} = \sum_{|\alpha|\leq k} \sup_{x\in\mathbb{R}^n} |\partial^\alpha u(x)| \leq C\|u\|_{H^s} . \tag{2.7.1}$$

- The fact that H^s functions can be approximated by $C_0^\infty(\mathbb{R}^n)$ functions means that for many purposes calculations can be done with $C_0^\infty(\mathbb{R}^n)$, or $\mathcal{S}(\mathbb{R}^n)$, functions and in the end the result extended to H^s. See the second worked problem to see how this goes to prove (2.7.1) with $k = 0$.

- For $s = 0$ we have $H^0 = L^2$ and the norm $\|\cdot\|_{H^0}$ is exactly the L^2 norm, while $\tilde{\|}\cdot\tilde{\|}_{H^0}$ is proportional (and hence equivalent) to the L^2 norm by the Parseval-Plancherel theorem.

- Strictly speaking the assertion $u \in C^k(\mathbb{R}^n)$ in the last sentence of the theorem only holds after possibly redefining u on a set of zero measure. This subtle point, which will generally be ignored in the following, arises because u is really only a distribution which can be represented by an L^2 function, and as such is only defined up to sets of zero measure.

On $(\mathbb{R}/(2\pi\mathbb{Z}))^n$: In the 2π-periodic case the following definitions are equivalent:

$$\begin{aligned} H^s_{per}([-\pi,\pi]^n) &= \{u \in L^2([-\pi,\pi]^n) : \|u\|^2_{H^s} = \sum_{\alpha:|\alpha|\le s} \|\partial^\alpha u\|^2_{L^2} < \infty\} \\ &= \{\sum_{m\in\mathbb{Z}^n} \hat{u}(m)e^{im\cdot x} : \sum_{m\in\mathbb{Z}^n} (1+\|m\|^2)^s |\hat{u}(m)|^2 < \infty\} \\ &= \overline{C^\infty_{per}([-\pi,\pi]^n)}^{\|\cdot\|_{H^s}} . \end{aligned}$$

Again the quantity apearing in the middle line defines an equivalent norm which can be used when it is more convenient. Since we are considering only the case $s = 0, 1, 2, \ldots$ the Fourier series $\sum_{m\in\mathbb{Z}^n} \hat{u}(m)e^{im\cdot x}$ always defines a square integrable function, and as s increases the function so defined is more and more regular (exercise), and as above we have:

Theorem 2.7.2 *For $s = 0, 1, 2\ldots$ the periodic Sobolev space $H^s_{per}([-\pi,\pi]^n)$ is a Hilbert space, and so complete in either of the norms*

$$\|u\|^2_{H^s} = \sum_{\alpha:|\alpha|\le s} \|\partial^\alpha u\|^2_{L^2} \quad \textit{or} \quad \tilde{\|u\|}^2_{H^s} = \sum_{m\in\mathbb{Z}^n} (1+\|m\|^2)^s |\hat{u}(m)|^2$$

which are equivalent. Given any $u \in H^s_{per}([-\pi,\pi]^n)$ there exists a sequence u_ν of $C^\infty_{per}([-\pi,\pi]^n)$ functions such that $\|u-u_\nu\|_{H^s} \to 0$ as $\nu \to +\infty$. If $u \in H^s([-\pi,\pi]^n)$ for $s > \frac{n}{2} + k$ with $k \in \mathbb{Z}_+$ then $u \in C^k(\mathbb{R}^n)$ and there exists $C > 0$ such that

$$\|u\|_{C^k} = \sum_{|\alpha|\le k} \sup_{x\in[-\pi,\pi]^n} |\partial^\alpha u(x)| \le C\|u\|_{H^s} . \tag{2.7.2}$$

Similar comments to those made after Theorem 2.7.1 apply of course. To keep the notation clean we do not indicate "periodic" in the notation for *norm* on a space of periodic functions, only the space - it should be clear from the context.

These definitions require some modifications for the case of general domains Ω, starting with the notion of the weak partial derivative (since we did not define distributions in Ω).

Definition 2.7.3 *A locally integrable function u defined on an open set Ω admits a weak partial derivative corresponding to the multi-index α if there exists a locally integrable function, designated $\partial^\alpha u$, with the property that*

$$\int_\Omega u\, \partial^\alpha \chi\, dx = (-1)^{|\alpha|} \int_\Omega \partial^\alpha u\, \chi\, dx ,$$

for every $\chi \in C^\infty_0(\Omega)$.

A useful fact is that in this situation:

$$\|\partial^\alpha u\|_{L^2} = \sup\{\int_\Omega u\,\partial^\alpha\chi\,dx \,:\, \chi\in C_0^\infty(\Omega) \text{ and } \|\chi\|_{L^2}=1\,.\} \tag{2.7.3}$$

Then employing this notion of partial derivative we define (for $s=0,1,2,\dots$):

$$H^s(\Omega)=\{u\in L^2(\Omega):\ \|u\|_{H^s}^2=\sum_{\alpha:|\alpha|\le s}\|\partial^\alpha u\|_{L^2}^2<\infty\}$$

(with all L^2 norms being defined by integration over Ω). This space is to be distinguished from the corresponding closure of the space of smooth functions supported in a compact subset of Ω:

$$H_0^s(\Omega)=\overline{C_0^\infty(\Omega)}^{\|\cdot\|_{H^s}}\,.$$

Since these functions are limits of functions which vanish in a neighbourhood of Ω they are to be thought of as vanishing in some generalized sense on $\partial\Omega$ (at least in the case $s=1,2,\dots$ and if Ω has a smooth boundary $\partial\Omega$.) The case $s=1$ gives the space $H_0^1(\Omega)$ which is the natural Hilbert space to use in order to give a weak formulation of the Dirichlet problem for the elliptic equation $Pu=f$ on Ω.

In one dimension, with $\Omega=(a,b)\subset\mathbb{R}$ the relation between $H^1((a,b))$ and $H_0^1((a,b))$ can be stated simply because all $f\in H^1((a,b))$ are uniformly continuous (similar to the argument in the last part of the 2nd worked problem) and

$$H_0^1((a,b))=\{f\in H^1((a,b)):f(a)=f(b)=0\}\,.$$

More details and proofs can be found in the relevant chapter of the book of Brezis. In this course we need to be able to use Sobolev spaces to study pde, and we will see that the H^s spaces are easy to work with for various reasons:

1. The "energy" methods often give rise to information about $\int u^2dx$ or $\int\|\nabla u\|^2dx$ where u is a solution of a pde, and this translates into information about the solution in Sobolev norms. As a specific example: conservation of energy

   $$\frac12\int(u_t^2+\|\nabla u\|^2)\,dx = E = constant\,,$$

   when u is a solution of the wave equation .

2. The Parseval-Plancherel theorem means that information on Sobolev norms is often easily obtainable when the solution is written down using Fourier methods.

3. The Sobolev spaces H^s are Hilbert spaces (complete) whose elements can be approximated by smooth functions: in practice, this means one has the dual advantages of smoothness of the functions and completeness of the space of functions.

Thus typically we will do some computations for smooth solutions of pde which give information about their Sobolev norms, and then using density we will extend the information to more general (weak) solutions lying in the Sobolev spaces themselves. The use of the full Sobolev space is crucial in any argument relying on completeness, typically in proving existence of a solution e.g. by variational methods or by the Lax-Milgram lemma.

2.8 Existence theorem for ordinary differential equations (ode)

We first note the following result:

Theorem 2.8.1 (Corollary to the contraction mapping principle) *Let* (X, d) *be a complete, non-empty, metric space and suppose* $T : X \to X$ *is such that* T^n *is a contraction for some* $n \in \mathbb{N}$. *Then* T *has a unique fixed point in* X; *in fact if* $y_0 \in X$, *then* $T^m y_0$ *converges to a fixed point as* $m \to \infty$.

Proof By Theorem 2.3.2, T^n has a unique fixed point, y. We also have that

$$T^n(Ty) = T^{n+1}y = T(T^n y) = Ty.$$

So Ty is also a fixed point of T^n and fixed points are unique so $Ty = y$. Also $T^{mn}y_0 \to y$ implies that $T^{mn+1}y_0 \to Ty = y$ and so on, until $T^{mn+(n-1)}y_0 \to y$ as $(m \to \infty)$. All together this implies that $T^m y_0 \to y$. □

Theorem 2.8.2 (Existence theorem for ode) *Let* $f(t, x)$ *be a vector-valued continuous function defined on the region*

$$\{(t, x) : |t - t_0| \leq a, \|x - x_0\| \leq b\} \subset \mathbb{R} \times \mathbb{R}^n$$

which also satisfies a Lipschitz condition in x:

$$\|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|.$$

Define $M = \sup |f(t, x)|$ *and* $h = \min(a, \frac{b}{M})$. *Then the differential equation*

$$\frac{dx}{dt} = f(t, x), \quad x(t_0) = x_0 \tag{2.8.1}$$

has a unique solution for $|t - t_0| \leq h$.

Proof This is a consequence of the contraction mapping theorem applied in the complete metric space:

$$X = \{x \in C([t_0 - h, t_0 + h], \mathbb{R}^n) : \|x(t) - x_0\| \leq Mh \; \forall t \in [t_0 - h, t_0 + h]\},$$

endowed with the metric $d(x_1, x_2) = \sup_{|t-t_0|\le h} \|x_1(t) - x_2(t)\|$. (Recall that a limit in the uniform norm of continuous functions is itself continuous.)

Introduce the integral operator T by the formula

$$(Tx)(t) = x_0 + \int_{t_0}^{t} f(s, x(s))ds. \tag{2.8.2}$$

The condition $Mh \le b$ implies that $T : X \to X$. Notice that $x \in X$ solves (2.8.1) if and only if $Tx = x$, by the fundamental theorem of calculus. In particular, observe that $Tx = x$ implies that $x \in X$ is in fact continuously differentiable.

We now assert that, for $|t - t_0| \le h$,

$$\|T^k x_1(t) - T^k x_2(t)\| \le \frac{L^k}{k!}|t - t_0|^k d(x_1, x_2).$$

For $k = 0$, this is obvious, and in general it follows by induction since

$$\begin{aligned} \|T^k x_1(t) - T^k x_2(t)\| &\le \int_{t_0}^t \|f(s, T^{k-1}x_1(s)) - f(s, T^{k-1}x_2(s))\|ds \\ &\le L\int_{t_0}^t \|T^{k-1}x_1(s) - T^{k-1}x_2(s)\|ds \\ &\le \tfrac{L^k}{(k-1)!}\int_{t_0}^t |s - t_0|^{k-1}\, ds\, d(x_1, x_2) \\ &\le \tfrac{L^k}{k!}|t - t_0|^k d(x_1, x_2). \end{aligned}$$

This implies that T^k is a contraction mapping for sufficiently large k, and so the result follows from Theorem 2.8.1. □

Theorem 2.8.3 (Gronwall Lemma) *Let $u \in C([t_0, t_1])$ satisfy*

$$u(t) \le A + B\int_{t_0}^t u(s)ds$$

for $t_0 \le t \le t_1$ and some positive constants A, B. Then

$$u(t) \le Ae^{B(t-t_0)}, \quad \text{for } t_0 \le t \le t_1 .$$

Proof Define $F(t) = A + B\int_{t_0}^t u(s)ds$ - by the fundamental theorem of calculus this is a C^1 function which verifies $\dot{F} = Bu \le BF$. It follows that $e^{-Bt}F(t)$ is non-increasing, so that $e^{-Bt}F(t) \le e^{-Bt_0}F(t_0)$, and hence

$$u(t) \le F(t) \le F(t_0)e^{B(t-t_0)} = Ae^{B(t-t_0)}$$

for $t_0 \le t \le t_1$. □

Theorem 2.8.4 (Well-posedness for ode) *In the situation of Theorem 2.8.2, let $y(t), w(t)$ be two solutions of (2.8.1) defined for $t_0 \le t \le t_1 \le t_0 + a$ and such that $\|y(t) - x_0\| \le b$ and $\|w(t) - x_0\| \le b$ on $[t_0, t_1]$. Then*

$$\|y(t) - w(t)\| \le \|y(t_0) - w(t_0)\|e^{L(t-t_0)}$$

for $t_0 \le t \le t_1$.

Proof $u(t) = y(t) - w(t)$ satisfies, by the Lipschitz property:

$$\|u(t)\| = \|y(t) - w(t)\| = \|y(t_0) - w(t_0) + \int_{t_0}^{t} \big(f(s, y(s)) - f(s, w(s))\big) ds\ \|$$
$$\leq \|u(t_0)\| + L \int_{t_0}^{t} \|u(s)\| ds\,.$$

The result is now an immediate consequence of Gronwall's inequality. □

2.9 Appendix: integration

The aim of this appendix[3] is to give a brief review of facts from integration needed - completeness of the L^p spaces, dominated convergence and other basic theorems. We first consider the case of functions on the unit interval $[0, 1]$. A main achievement of the Lebesgue integral is to construct *complete* vector spaces of functions where the completeness is with respect to a norm defined by an integral such as the L^2 norm $\|\cdot\|_{L^2}$ defined by

$$\|f\|_{L^2}^2 = \int_0^1 |f(x)|^2\, dx\,.$$

This is a perfectly good norm on the space of continuous functions $C([0, 1])$, but the resulting normed vector space is not complete (and so not a Banach space) and is not so useful as a setting for analysis. The Lebesgue framework provides a larger class of functions which can be potentially integrated - the *measurable functions*. The complete Lebesgue space L^2 which this construction leads to then consists of (equivalence classes of) measurable functions f with $\|f\|_{L^2}^2 < \infty$; here it is necessary to consider equivalence classes of functions because functions which are non-zero only on sets which are very small (in a certain precise sense) are invisible to the integral, and so have to be factored out of the discussion. The "very small" sets in question are called null sets and are now defined.

2.9.1 Null sets and measurable functions on $[0, 1]$

An interval in $[0, 1]$ is a subset of the form (a, b) or $[a, b]$ or $(a, b]$ or $[a, b)$ (respectively open,closed, half open). In all cases the length of the interval is $|I| = b - a$. A collection of intervals $\{I_\alpha\}$ *covers* a subset A if $A \subset \cup_\alpha I_\alpha$.

Definition 2.9.1 (Null sets) *For a set $A \subset [0, 1]$ we define the* outer measure *to be*

$$|A|_* = \inf_{\{I_n\}_{n=1}^{\infty} \in \mathcal{C}} \Big\{ \sum_n |I_n| \ :\ A \subset \cup I_n \Big\}\,,$$

[3] This section gives a brief introduction to the results on Lebesgue integral which we make use of. You should be able to use the results listed here but will not be examined on the proofs or on any subtleties connected with the results.

where $\mathcal{C}$ consists of countable families of intervals in $[0,1]$. *A set* $N \subset [0,1]$ *is null if* $|I|_* = 0$, *i.e. if for all* $\epsilon > 0$ *there exists* $\{I_n\}_{n=1}^{\infty} \in \mathcal{C}$ *which covers* A *with* $\sum |I_n| < \epsilon$.

Definition 2.9.2 *We say* $f = g$ *almost everywhere (a.e.) if* $f(x) = g(x)$ *for all* $x \notin N$ *for some null set* N. *We say a sequence of functions* f_n *converges to* f *a.e. if* $f_n(x) \to f(x)$ *for all* $x \notin N$ *for some null set* N.

Equality a.e. defines an equivalence relation, and two equivalent functions f, g are said to be Lebesgue or measure theoretically equivalent. One way to think about measurable functions is provided by the Lusin theorem, which says a measurable function is one which is "almost continuous" in the sense that it agrees with a continuous function on the complement of a set of arbitrarily small outer measure:

Definition 2.9.3 (Measurable functions) *A function* $f : [0,1] \to \mathbb{R}$ *is measurable if for every* $\epsilon > 0$ *there exists a continuous function* $f^{\epsilon} : [0,1] \to \mathbb{R}$ *and a set* F^{ϵ} *such that* $|F^{\epsilon}|_* < \epsilon$ *and* $f(x) = f^{\epsilon}(x)$ *for all* $x \notin F^{\epsilon}$. *We write* $L([0,1])$ *for the space of all measurable functions so defined.*

Theorem 2.9.4 $L([0,1])$ *is a linear space closed under almost everywhere convergence: given a sequence* $f_n \in L([0,1])$ *of measurable functions which converges to a function* f *a.e. it follows that* $f \in L([0,1])$.

Definition 2.9.3 is not the usual definition of measurability - which involves the notion of a distinguished collection of sets, the σ-algebra of measurable sets - but is equivalent to it by what is called the *Lusin theorem* (see for example §2.4 and §7.2 in the book *Real Analysis* by Folland). The Lusin theorem gives a helpful way of thinking about measurability (the Littlewood 3 principles - see §3.3 in the book *Real Analysis* by Royden and Fitzpatrick). A companion to the Lusin theorem is the *Egoroff theorem* which states that given a sequence $f_n \in L([0,1])$ of measurable functions which converges to a function f a.e. then for every $\epsilon > 0$ it is possible to find a set $E \subset [0,1]$ with $|E|_* < \epsilon$ such that $f_n \to f$ uniformly on $E^c = [0,1] - E$. Thus two of Littlewood's principles say that " a measurable function is one which agrees with a continuous function except on a set which may be taken to have arbitrarily small size" and "a sequence of measurable functions which converges almost everywhere converges uniformly on the complement of a set which may be assumed to be arbitrarily small".

2.9.2 Definition of $L^p([0,1])$

An integral $\int_0^1 f(x)dx$ can be defined of any non-negative measurable function, although the value can be $+\infty$. When the function is continuous, or indeed Riemann integrable, this integral agrees with the Riemann integral, and it has the following properties (for arbitrary non-negative measurable functions f, g):

1. $\int_0^1 cf(x)dx = c\int_0^1 f(x)dx$ if $c > 0$,

2. $\int_0^1 (f(x) + g(x))\, dx = \int_0^1 (f(x) + g(x))\, dx$,

3. $\int_0^1 f(x)\, dx \leq \int_0^1 g(x)\, dx$ if $f \leq g$ a.e.

Exercise For non-negative measurable functions f, g, show that if $f = g$ a.e. then $\int_0^1 f(x)dx = \int_0^1 g(x)dx$.

Accepting that such a definition exists, we can now define the $L^p([0,1])$ spaces, which are Banach spaces of functions on which there exists a well-defined notion of the integral (called the Lebesgue integral).

Definition 2.9.5 *For* $1 \leq p < \infty$ *define* $L^p([0,1])$ *to be the linear space of measurable functions on* $[0,1]$ *with the property that*

$$\|f\|_{L^p}^p = \int_0^1 |f(x)|^p\, dx < \infty .$$

For the case $p = \infty$: firstly, say that f is essentially bounded above with upper (essential) bound M if $f(x) \leq M$ for $x \notin N$ for some null set N. Then let ess sup f be the infimum of all upper essential bounds. Then:

Definition 2.9.6 $L^\infty([0,1])$ *is the linear space of measurable functions on* $[0,1]$ *with the property that*

$$\|f\|_{L^\infty} = \text{ess sup } |f| < \infty .$$

A crucial fact for us is that considering the spaces of equivalence classes of functions which agree almost everywhere we obtain *Banach* spaces, also written $L^p([0,1])$: these "Lebesgue spaces" are vector spaces of (equivalence classes of) functions which are *complete* with respect to the norm $\|\cdot\|_{L^p}$. (The fact that, strictly speaking, the elements of these spaces are equivalence classes of functions which agree almost everywhere, is often taken as understood and not repeatedly mentioned each time the spaces are made use of.)

The spaces $L^p([0,1])$ contain the continuous functions, and the Lebesgue integral, which is defined on the whole of these spaces, is equal to the Riemann integral when restricted to Riemann integrable functions. These $L^p([0,1])$ spaces are special cases of $L^p(\mathcal{M})$ spaces which arise from abstract measure spaces $\mathcal{M}$ on which a measure μ (and a σ-algebra of measurable sets) is given; μ measures the "size" of elements of this collection of measurable sets. In the general setting the integral of a function is often defined in terms of the measure of sets on which the function takes given values: for example, one development of the integral takes as starting point the following definition for the integral of a non-negative measurable function:

$$\int f\, d\mu = \int_0^\infty \mu(\{f > \lambda\})\, d\lambda . \tag{2.9.1}$$

The point here is that as λ increases, the sets $\{f > \lambda\}$ decrease and their measure $\mu(\{f > \lambda\})$ decreases also, so that (2.9.1) is well-defined as the Riemann integral of a monotone function. See the book *Analysis* by Lieb and Loss for a development along these lines.

Other examples of measure spaces used in this course are

- $L^p([a,b])$ with norm $(\int_a^b |f(x)|^p\,dx)^{\frac{1}{p}}$,
- $L^p([-\pi,\pi]^n)$ with norm $(\int_{[-\pi,\pi]^n} |u(x)|^p\,dx)^{\frac{1}{p}}$, and
- $L^p(\Omega)$ with norm $(\int_\Omega |f(x)|^p\,dx)^{\frac{1}{p}}$, where $\Omega \subset \mathbb{R}^n$ is open; the case $\Omega = \mathbb{R}^n$ will occur most often.

2.9.3 Assorted theorems on integration

Theorem 2.9.7 (Hölder inequality) *$\int fg dx \le \|f\|_{L^p}\|g\|_{L^q}$ for any pair of functions $f \in L^p, g \in L^q$ (on any measure space) with $p^{-1} + q^{-1} = 1$ and $p, q \in [1,\infty]$.*

Corollary 2.9.8 (Young inequality) *If $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$ then $f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_{L^p} \le \|f\|_{L^p}\|g\|_{L^1}$ for $1 \le p \le \infty$.*

Theorem 2.9.9 (Dominated convergence theorem) *Let the sequence $f_n \in L^1$ converge to f almost everywhere (on any measure space) and assume that there exists a nonnegative measurable function $\Phi \ge 0$ such that $|f_n(x)| \le \Phi(x)$ almost everywhere and $\int \Phi < \infty$. Then $\lim_{n\to\infty} \int f_n = \int f$ and $\lim_{n\to\infty} \|f_n - f\|_{L^1} = 0$.*

Corollary 2.9.10 (Differentiation through the integral) *Let $g \in C^1(\mathbb{R}^n \times \Omega)$ where $\Omega \subset \mathbb{R}^m$ is open, and consider $F(\lambda) = \int_{\mathbb{R}^n} g(x,\lambda)dx$. Assume there exists a measurable function $\Phi(x) \ge 0$ such that*

- $\int_{\mathbb{R}^n} \Phi(x)\,dx < \infty$,
- $\sup_\lambda(|g(x,\lambda) + |\partial_\lambda g(x,\lambda)|) \le \Phi(x)$.

Then $F \in C^1(\Omega)$ and $\partial_\lambda F = \int_{\mathbb{R}^n} \partial_\lambda g(x,\lambda)\,dx$.

Corollary 2.9.11 *If f is a $C^k(\mathbb{R}^n)$ function with all partial derivatives $\partial^\alpha f$ of order $|\alpha| \le k$ bounded, and $g \in L^1(\mathbb{R}^n)$ then $f*g \in C^k(\mathbb{R}^n)$ and $\partial^\alpha(f*g) = (\partial^\alpha f)*g$ for $|\alpha| \le k$.*

Theorem 2.9.12 (Tonelli) *If $f \ge 0$ is a nonnegative measurable function $f : \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$ then*

$$\iint\limits_{\mathbb{R}^l\times\mathbb{R}^m} f(x,y)\,dxdy = \int\limits_{\mathbb{R}^l}\left(\int\limits_{\mathbb{R}^m} f(x,y)\,dy\right)dx = \int\limits_{\mathbb{R}^m}\left(\int\limits_{\mathbb{R}^l} f(x,y)\,dx\right)dy\,.$$

Theorem 2.9.13 (Fubini) *If f is a measurable function $f : \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$ such that*

$$\iint_{\mathbb{R}^l \times \mathbb{R}^m} |f(x,y)|\, dxdy < \infty$$

then

$$\iint_{\mathbb{R}^l \times \mathbb{R}^m} f(x,y)\, dxdy = \int_{\mathbb{R}^l} \left(\int_{\mathbb{R}^m} f(x,y)\, dy \right) dx = \int_{\mathbb{R}^m} \left(\int_{\mathbb{R}^l} f(x,y)\, dx \right) dy\,.$$

Remark 2.9.14 *In these two results it is to be understood that when we write down repeated integrals that an implicit assertion is that the functions $y \mapsto \int f(x,y)dx$ and $x \mapsto \int f(x,y)dy$ are measurable and integrable.*

Theorem 2.9.15 (Minkowski inequality) *If f is a measurable function $f : \mathbb{R}^l \times \mathbb{R}^m \to \mathbb{R}$ and $g : \mathbb{R}^m \to \mathbb{R}$ is measurable, then*

$$\| \int_{\mathbb{R}^m} f(x,y)g(y)\, dy \|_{L^p(dx)} \le \int_{\mathbb{R}^m} \|f(x,y)\|_{L^p(dx)} |g(y)|\, dy\,. \tag{2.9.2}$$

where

$$\|f(x,y)\|^p_{L^p(dx)} = \int_{\mathbb{R}^l} |f(x,y)|^p\, dx\,,$$

with the understanding as above that this means that if the right hand side of (2.9.2) *is finite then the function $f(x,y)g(y)$ is integrable in y for almost every x and the resulting function $x \mapsto \int f(x,y)g(y)\, dy$ is measurable and* (2.9.2) *holds.*

2.10 Worked problems

1. Prove that if a continuous 2π-periodic function $f \in C_{per}([-\pi,\pi])$ satisfies

$$\hat{f}(m) = (2\pi)^{-1} \int_{-\pi}^{+\pi} e^{-imx} f(x) dx = 0$$

for all $m \in \mathbb{Z}$, then f is identically zero. Deduce that if $f \in C^\infty_{per}([-\pi,\pi])$ then $f = \sum_{m\in\mathbb{Z}} \hat{f}(m) e^{imx}$.

Answer Assume for the sake of contradiction that there exists x_0 with $f(x_0) \neq 0$. Replacing $f(\cdot)$ by $\pm f(\cdot - x_0)$ we may assume w.l.o.g. that $f(0) > 0$. Now:

$$\int_{-\pi}^{+\pi} f(x)(\epsilon + \cos x)^k\, dx = 0$$

for all $k \in \mathbb{Z}_+$ and $\epsilon \in \mathbb{R}$, by the assumption that all Fourier coefficients vanish. By continuity of f there exists $\delta \in (0, \pi/2]$ such that $f(x) > f(0)/2 > 0$ for $|x| < \delta$. Now $\max_{\delta \le |x| \le \pi} \cos x < 1$ and $\cos 0 = 1$, so there exists

- $\epsilon > 0$ such that $|\epsilon + \cos x| < 1 - \epsilon/2$ for $\delta \leq |x| \leq \pi$;
- $\eta \in (0, \delta)$ such that $|\epsilon + \cos x| > 1 + \epsilon/2$ for $|x| < \eta$.

Now

$$\begin{aligned}\int_{-\pi}^{+\pi} f(x)(\epsilon + \cos x)^k \, dx &= \int_{|x|\leq\eta} + \int_{|\eta|<|x|<\delta} + \int_{\delta\leq|x|\leq\pi} f(x)(\epsilon + \cos x)^k \, dx \\ &\geq (1 + \frac{\epsilon}{2})^k \frac{f(0)}{2} - 2\pi \sup |f| (1 - \frac{\epsilon}{2})^k ,\end{aligned}$$

since the middle integral is ≥ 0 because $f > 0$ on $(-\delta, +\delta)$, and also since $\delta \leq \pi/2$ and $\cos x \geq 0$ on $[0, \pi/2]$. Now let $k \to +\infty$: the final term has limit zero, while the first term has limit $+\infty$ providing a contradiction.

For the last part observe that for $f \in C_{per}([-\pi, \pi])$ the Fourier coefficients satisfy

$$\sup_m m^N |\hat{f}(m)| < \infty$$

for all N (rapidly decreasing) and therefore the series $\sum_{m\in\mathbb{Z}} \hat{f}(m)e^{imx}$ converges absolutely to define a continuous periodic function whose Fourier coefficients are $\hat{f}(m)$. (The latter assertion follows from the fact that the sum and integral can be interchanged when integrating an absolutely and uniformly convergent power series.) Therefore $f(x) - \sum_{m\in\mathbb{Z}} \hat{f}(m)e^{imx}$ is a continuous 2π-periodic function whose Fourier coefficents all vanish. It therefore vanishes itself by the previous part, completing the proof.

2. For positive s the Sobolev space is defined as

$$H^s(\mathbb{R}^n) = \{f \in L^2(\mathbb{R}^n) : \|f\|^2_{H^s} = \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^s |\hat{f}(\xi)|^2 \, d\xi < \infty\} .$$

Show that if $s > n/2$ then $H^s(\mathbb{R}^n) \subset C(\mathbb{R}^n)$ and there exists a positive number C such that

$$\sup_{x\in\mathbb{R}^n} |f(x)| \leq C\|f\|_{H^s} . \qquad (2.10.1)$$

In the case $n = 1$ prove, using calculus only, the inequality

$$\sup_{x\in\mathbb{R}} |f(x)| \leq C' \left(\int (f^2 + (\partial_x f)^2) \, dx \right)^{\frac{1}{2}} . \qquad (2.10.2)$$

for all $f \in \mathcal{S}(\mathbb{R})$ and for some positive C'. Comment on the relation with the first part of the question. Prove that all functions $f \in H^1(\mathbb{R})$ are uniformly continuous on $\mathbb{R}$.

Answer We will first establish the inequality for $f \in \mathcal{S}(\mathbb{R}^n)$. By the Hölder inequality:

$$\begin{aligned}(2\pi)^n |\mathcal{F}^{-1}(\hat{f})| = \left| \int_{\mathbb{R}^n} \hat{f}(\xi)e^{i\xi\cdot x} \, dx \right| &= \left| \int_{\mathbb{R}^n} \frac{(1 + \|\xi\|^2)^{\frac{s}{2}} \hat{f}(\xi)e^{i\xi\cdot x}}{(1 + \|\xi\|^2)^{\frac{s}{2}}} \, dx \right| \\ &\leq \left(\int_{\mathbb{R}^n} \frac{1}{(1 + \|\xi\|^2)^s} \, dx \right)^{\frac{1}{2}} \|f\|_{H^s} .\end{aligned}$$

The integral on the second line is finite, since adopting polar coordinates (r, Ω) it is just

$$\int_{S^{n-1}} d\Omega \int_0^\infty \frac{r^{n-1}}{(1+r^2)^s} dr$$

which is finite for $2s - n + 1 > 1$, i.e. for $s > n/2$. This establishes the stated inequality (2.10.1). Exactly the same calculation shows that $\|\hat{f}\|_{L^1} \le C\|f\|_{H^s}$. Now to complete the proof, just approximate $f \in H^s(\mathbb{R}^n)$ by a sequence f_ν of Schwartz functions as in Theorem 2.7.1: since the constant in (2.10.1) is independent of ν we can take the limit $\nu \to \infty$. Then since $\|f - f_\nu\|_{H^s} \to 0$ the sequence f_ν is Cauchy in the norm H^s, and hence also Cauchy in the uniform C^0 norm by (2.10.1). This implies that the limit $f \in C^0(\mathbb{R}^n)$ and obeys (2.10.1).

For the one dimensional inequality calculate, by the Hölder inequality that

$$|f(x) - f(y)| = |\int_y^x f'(z)dz| \le |x-y|^{\frac{1}{2}} \left| \int (f')^2 dz \right|^{\frac{1}{2}}$$

and

$$|f^2(x) - f^2(y)| = |2\int_y^x f(z)f'(z)dz| \le \int_{\mathbb{R}} f^2 + (f')^2\, dz\,.$$

For $f \in \mathcal{S}(\mathbb{R})$ let $y \to +\infty$ in the second inequality and the result follows for such f. It holds for general $f \in H^1(\mathbb{R})$ by density (strictly speaking up to sets of measure zero). The first inequality implies uniform continuity.

Relation with the first part of the question: the Parseval-Plancherel theorem implies that:

$$\int_{\mathbb{R}} f^2 + (f')^2\, dz = \frac{1}{2\pi}\int_{\mathbb{R}} (|\hat{f}(\xi)|^2 + |\xi|^2|\hat{f}(\xi)|^2)\, d\xi\,.$$

So that the result of the second part is really a special case of the first part, but the proof is different.

2.11 Example sheet 2

1. Obtain and solve the ODE satisfied by characteristic curves $y = y(x)$ for the equation $(x^2+2)^2 u_{xx} - (x^2+1)^2 u_{yy} = 0$. Show that there are two families of such curves which can be written in the form $y - x + 2^{-\frac{1}{2}} \arctan 2^{-\frac{1}{2}} x = \xi$ and $y + x - 2^{-\frac{1}{2}} \arctan 2^{-\frac{1}{2}} x = \eta$, for arbitrary real numbers ξ, η. Now considering the change of coordinates $(x, y) \to (\xi, \eta)$ so determined find the form of the equation in the coordinate system (ξ, η).

2. Which of the following functions of x lie in Schwartz space $\mathcal{S}(\mathbb{R})$: (a) $(1+x^2)^{-1}$,(b) e^{-x}, (c)$e^{-x^4}/(1+x^2)$? Show that if $f \in \mathcal{S}(\mathbb{R})$ then so is $f(x)/P(x)$ where P is any strictly positive polynomial (i.e. $P(x) \ge \theta > 0$ for some real θ.

3. Solve the following initial value problem

$$\partial_t u = \partial_x^3 u \qquad u(0,x) = f(x)$$

for $x \in [-\pi, \pi]$ with periodic boundary conditions $u(t, -\pi) = u(t, \pi)$ and f smooth and 2π-periodic. Discuss well-posedness properties of your solutions for the L^2 norm, i.e. $\|u(t)\|_{L^2} = (\int_{-\pi}^{+\pi} |u(t,x)|^2 dx)^{\frac{1}{2}}$, using the Parseval-Plancherel theorem.

4. Show that the heat equation $\partial_t u = \partial_x^2 u$, with 2π-periodic boundary conditions in x, is well-posed forwards in time in L^2 norm, but not backwards in time (even locally). (Hint compute the L^2 norm of solutions u_n for negative t corresponding to initial values $u_n(0,x) = n^{-1}e^{inx}$.)

5. (i) Use Fourier series to solve the Schrödinger equation

$$\partial_t u = i\partial_x^2 u \qquad u(0,x) = f(x)$$

for initial value f smooth and periodic. Prove in two different ways that there is only one smooth periodic solution.

(ii) Use the fourier transform to solve the Schrodinger equation for $x \in \mathbb{R}$ and initial value $f \in \mathcal{S}(\mathbb{R})$. Find the solution for the case $f = e^{-x^2}$.

6. (i) Verify that the tempered distribution u on the real line defined by the function $(2m)^{-1}e^{-m|x|}$, (for positive m), solves

$$(\frac{-d^2}{dx^2} + m^2)u = \delta_0$$

in $\mathcal{S}'(\mathbb{R})$.

(ii) Verify that the function on the real line $g(x) = 1$ for $x \le 0$ and $g(x) = e^{-x}$ for $x > 0$ defines a tempered distribution T_g which solves in $\mathcal{S}'(\mathbb{R})$

$$T'' + T' = -\delta_0.$$

7. (a) Write down the precise distributional meaning of the equation

$$-\Delta(|x|^{-1}) = 4\pi\delta_0 \qquad \text{in } \mathcal{S}'(\mathbb{R}^3)$$

in terms of test functions, and then use the divergence theorem to verify that it holds. (Hint: apply the divergence theorem on the region $\{0 < |x| < R\} - \{0 < |x| < \epsilon\}$ for R sufficiently large and take the limit $\epsilon \to 0$ carefully).

(b) Find the fundamental solution $K_m \in \mathcal{S}'(\mathbb{R}^3)$ of the operator $(-\Delta + m^2)$ with $m > 0$ and in the case of domain $\mathbb{R}^3$. Indicate the modifications of (a) required to prove this.

8. Define, for non-negative s, the norm $\|\cdot\|_s$ on the space of smooth 2π-periodic function of x by

$$\|f\|_s^2 \equiv \sum_{m\in\mathbb{Z}} (1+m^2)^s |\hat{f}(m)|^2$$

where $\hat{f}(n)$ are the fourier coefficients of f. (This is called the Sobolev H^s norm).

(i) What are these norms if $s = 0$? Write down a formula for these norms for $s = 0, 1, 2\ldots$ in terms of $f(x)$ and its derivatives directly. (Hint Parseval).

(ii) If $u(t,x)$ is the solution for the heat equation with 2π-periodic boundary conditions, then for $t > 1$ and $s = 0, 1, 2, , ,$ find a number $C_s > 0$ such that

$$\|u(t,\cdot)\|_s \le C_s \|u(0,\cdot)\|_0 \; .$$

(iii) Show that there exists a number $\gamma_1 > 0$ which does not depend on f so that $\max|f(x)| \le \gamma_1\|f\|_1$ for all smooth 2π-periodic f. For which $s > 0$ is it also true that there exists $\gamma_s > 0$ such that $\max|f(x)| \le \gamma_s\|f\|_s$ for all smooth 2π-periodic functions f?

(iv) Generalize the inequality in the last sentence of (iii) to periodic functions $f = f(x_1, \ldots x_n)$ of n variables. Find a number $\sigma(n)$ such that the inequality holds if and only if $s > \sigma(n)$?

9. For each of the following equations, find the most general tempered distribution T which satisfies it.

$$xT = 0, \;\; xdT/dx = 0, \;\; x^2T = \delta_0, \;\; xdT/dx = \delta_0$$

$$dT/dx = \delta_0, \;\; dT/dx + T = \delta_0 \;\; T - (d/dx)^2T = \delta_0.$$

(Hint: see Friedlander §2.7).

10. Solve the equation $x^mT = 0$ in $\mathcal{S}'(\mathbb{R})$.

3 Elliptic equations

3.1 Introduction and Notation

The equation

$$-\Delta u + u = f \tag{3.1.1}$$

can be solved for u via the Fourier transform, if $f \in \mathcal{S}(\mathbb{R}^n)$. The solution is the inverse Fourier transform of

$$\hat{u}_f(\xi) = \frac{\hat{f}(\xi)}{1 + \|\xi\|^2}\,; \tag{3.1.2}$$

this formula defines a Schwartz function, and hence the solution $u = u_f \in \mathcal{S}$ also, and the mapping $f \mapsto u_f$ is continuous in the sense that if f_n is a sequence of Schwartz functions such that $\|f_n - f\|_{\alpha,\beta} \to 0$ for every Schwartz semi-norm $\|\cdot\|_{\alpha,\beta}$, then also $\|u_n - u\|_{\alpha,\beta} \to 0$ for every Schwartz semi-norm, where $u_n = u_{f_n}$, $u = u_f$.

In fact the formula above extends to define a distributional solution u_f for each tempered distribution $f \in \mathcal{S}'(\mathbb{R}^n)$, i.e. for each $\phi \in \mathcal{S}(\mathbb{R}^n)$ there holds

$$\langle\, u_f\,,\, -\Delta\phi + \phi\,\rangle = \langle\, f\,,\, \phi\,\rangle\,.$$

Using the Fourier transform definition of the Sobolev space one can check that:

$$\|u_f\|^2_{H^{s+2}} = \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^{s+2}|\hat{u}(\xi)|^2\, d\xi = \int_{\mathbb{R}^n} (1 + \|\xi\|^2)^{s}|\hat{f}(\xi)|^2\, d\xi = \|f\|^2_{H^s}\,.$$

Thus the solution operator

$$\begin{aligned} (-\Delta + 1)^{-1} \,:\, H^s &\to H^{s+2} \\ f &\mapsto u_f \end{aligned}$$

is bounded, indicating that the solution gains two derivatives, *as measured in* L^2, compared to the inhomogeneous term. This phenomenon goes under the name *elliptic regularity*, and generalizes to wide classes of elliptic equations, as does the *maximum principle* bound

$$\max_{x\in\mathbb{R}^n} |u_f(x)| \leq \max_{x\in\mathbb{R}^n} |f(x)| , \tag{3.1.3}$$

which is valid for classical (e.g. Schwartz) solutions, and is an immediate consequence of the calculus necessary conditions for $u \in \mathcal{S}(\mathbb{R}^n)$ to attain a maximum/minimum at a point x_*:

$$\begin{aligned} \partial_j u(x_*) = 0\,, \quad \partial^2 u(x_*) \geq 0\,, \qquad &(\text{minimum})\,; \\ \partial_j u(x_*) = 0\,, \quad \partial^2 u(x_*) \leq 0\,, \qquad &(\text{maximum})\,. \end{aligned}$$

The notation indicates definiteness of the symmetric matrices $\partial^2 u(x_*) = \partial^2_{jk} u(x_*)$. This definiteness implies that at a maximum $\Delta u(x_*) = Tr\, \partial^2 u(x_*) \leq 0$ and hence by (3.1.1) that $\max u = u(x_*) \leq f(x_*) \leq \max |f|$; a similar argument for the minimum completes the derivation of (3.1.3) for $\mathcal{S}(\mathbb{R}^n)$ solutions. It is clear from the proof just outlined that this result is generalizable, both to more general classical solutions and also to larger classes of equations.

It is the purpose of this chapter to explain the generalizations of the results just discuused from (3.1.1) to much larger classes of second order elliptic equations.

*Notation:*Let $B_R = \{w : |w| < R\}$ and $\overline{B_R} = \{w : |w| \leq R\}$ be the open and closed balls of radius R and more generally let $B_R(x) = \{w : |w - x| < R\}$ and $\overline{B_R(x)} = \{w : |w-x| \leq R\}$. We write $\partial B_R, \partial B_R(x)$ for the corresponding spheres, i.e. $\partial B_R(x) = \{w : |w - x| = R\}$ etc. In the following $\Omega \subset \mathbb{R}^n$ is always open and bounded unless otherwise stated, $\overline{\Omega}$ is its closure and $\partial\Omega$ is its boundary (always assumed smooth).

3.2 Existence of solutions

In this section it is explained how to formulate and solve elliptic boundary value problems via the Lax-Milgram lemma, starting with the case of periodic boundary conditions.

Definition 3.2.1 *A weak solution of* $Pu = f \in L^2([-\pi,\pi]^n)$*, with* P *the operator given by*

$$Pu = -\sum_{jk} \partial_j (a_{jk} \partial_k u) + cu\,, \tag{3.2.1}$$

with smooth periodic coeffcients $a_{jk} = a_{kj} \in C^\infty([-\pi,\pi]^n)$ *and* $c \in C^\infty([-\pi,\pi]^n)$*, is a function* $u \in H^1_{per}([-\pi,\pi]^n)$ *with the property that*

$$\int \sum_{jk} a_{jk}\, \partial_j u \partial_k v + cuv\, dx = \int fv\, dx \tag{3.2.2}$$

for all $v \in H^1_{per}([-\pi,\pi]^n)$.

Theorem 3.2.2 *Let* P *be as in* (3.2.1)*, and assume that the inequalities*

$$m\|\xi\|^2 \le \sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \le M\|\xi\|^2 \tag{3.2.3}$$

and

$$c(x) \ge c_0 > 0 \tag{3.2.4}$$

hold everywhere, for some positive constants m, M, c_0 *and all* $\xi \in \mathbb{R}^n$*. Then given* $f \in L^2([-\pi,\pi]^n)$ *there exists a unique weak solution of* $Pu = f$ *in the sense of definition* (3.2.1).

Proof Define the bilinear form $B(u,v) = \int \sum_{jk} a_{jk}\,\partial_j u \partial_k v + cuv\,dx$ and observe that it obeys the continuity and coercivity conditions in the Lax-Milgram lemma in the Hilbert space H^1_{per}. In particular for continuity take $\|B\| = \|a\|_{L^\infty} + \|c\|_{L^\infty}$, where the norm for the matrix $a(x) = (a_{jk}(x))_{j,k=1}^n$ is the operator norm. For coercivity, notice that (3.2.3) and (3.2.4) imply

$$B(u,u) \;\ge\; \min\{m, c_0\}\,\|u\|^2_{H^1}\,. \tag{3.2.5}$$

The right hand side of (4.5.2) defines a bounded functional $L(v)$, since

$$|L(v)| = |\int fv dx| \le \|f\|_{L^2}\|v\|_{L^2} \le \|f\|_{L^2}\|v\|_{H^1}\,,$$

by the Hölder inequality, and so existence and uniqueness follows from the Lax-Milgram lemma. □

Definition 3.2.1 and theorem 3.2.2 have various generalizations: to obtain the correct definition of weak solution for a given elliptic boundary value problem the general idea is to start with a classical solution and multiply by a test function and integrate by parts using the boundary conditions in their classical format. This will lead to a weak formulation of both the equation and the boundary conditions. For example the weak formulation of the Dirichlet problem

$$Pu = f\,, \qquad u|_{\partial\Omega} = 0\,, \tag{3.2.6}$$

where

$$Pu = -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \sum_{j=1}^{n} b_j\partial_j u + cu \tag{3.2.7}$$

for continuous functions $a_{jk} = a_{kj}, b_j$ and c, is to find a function $u \in H^1_0(\Omega)$ such that

$$B(u,v) = L(v)\,, \quad \forall v \in H^1_0(\Omega)\,,$$

where $L(v) = \int f v\, dx$ (a bounded linear map/functional), and B is the bilinear form:

$$B(u,v) = \int \big(\sum_{jk} a_{jk}\, \partial_j u \partial_k v + \sum b_j \partial_j u v + cuv\big)\, dx\,.$$

By the Lax-Milgram lemma we have

Theorem 3.2.3 *In the situation just described, assume that* (3.2.3) *and* (3.2.4) *hold. Then if* $\|b\|_{L^\infty}$ *is sufficiently small, there exists a unique weak solution to* (3.2.6).

Proof The crucial point is that (3.2.5) changes into

$$B(u,u) \;\geq\; \min\{m, c_0\}\, \|u\|^2_{H^1} \;-\; \|b\|_{L^\infty} \|u\|^2_{H^1}\,, \tag{3.2.8}$$

where $\|b\|_{L^\infty} = \sup_x \|b(x)\| = \sup_x (\sum_{j=1}^n b_j(x)^2)^{\frac{1}{2}}$. The remainder of the proof is essentially as above. □

This solution has various regularity properties, the simplest of which is that if in addition $a_{jk} \in C^1(\Omega)$ then in any ball such that $\overline{B_r(y)} \subset \Omega$ there holds for some constant $C > 0$:

$$\|u\|_{H^2(B_r(y))} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)})\,, \qquad \text{(interior } H^2 \text{ regularity)}\,,$$

and if in addition all the coefficients are smooth then we have, for arbitrary $s \in \mathbb{N}$ and some $C_s > 0$:

$$\|u\|_{H^{s+2}(B_r(y))} \leq C_s(\|f\|_{H^s(\Omega)} + \|u\|_{L^2(\Omega)})\,, \qquad \text{(higher interior regularity)}\,.$$

For the periodic case there is no boundary, and these results hold with the balls $B_r(y)$ replaced by the whole domain of periodicity $[-\pi, \pi]^n$. For example, consider the Poisson equation

$$-\Delta\, u \;=\; f = \sum_{m \in \mathbb{Z}^n} \hat{f}(m)\, e^{im \cdot x} \tag{3.2.9}$$

with periodic boundary conditions.

Theorem 3.2.4 *(i) If* $u \in C^2_{per}$ *is a classical solution of* (3.2.9) *then necessarily* $\hat{f}(0) = 0$.

(ii) If $f \in L^2$ *and* $\hat{f}(0) = 0$*, then there is a unique weak solution of* (3.2.9) *in the Hilbert space*

$$H^1_{per,0} \;=\; \{u \in H^1_{per} \,:\, \hat{u}(0) = 0\}$$

given by

$$u(x) = \sum_{m \in \mathbb{Z}^n \setminus \{0\}} \frac{\hat{f}(m)}{\|m\|^2}\, e^{im \cdot x}\,. \tag{3.2.10}$$

Furthermore, this solution satisfies $\|u\|_{H^{s+2}} \leq 2\|f\|_{H^s}$ *whenever* f *also belongs to* H^s_{per}.

Proof $H^1_{per,0} \subset H^1_{per}$ is a closed subspace, and is thus a Hilbert space using the same inner product as H^1_{per}. The fact that $\frac{1}{2} \leq \frac{\|m\|^2}{1+\|m\|^2} \leq 1$ for all $m \in \mathbb{Z}^n \backslash \{0\}$ implies that

$$B(u,v) = \int \nabla u \cdot \nabla v \, dx = (2\pi)^n \sum_{m \in \mathbb{Z}^n \backslash \{0\}} \|m\|^2 \hat{u}(-m)\hat{v}(m)$$
$$= (2\pi)^n \sum_{m \in \mathbb{Z}^n \backslash \{0\}} \|m\|^2 \overline{\hat{u}(m)}\hat{v}(m)$$

satisfies the continuity and coercivity conditions in the Lax-Milgram lemma, applied in the Hilbert space $H^1_{per,0}$. A weak solution in this space means a function $u \in H^1_{per,0}$ such that $B(u,v) = \int f v dx$ for all $v \in H^1_{per,0}$; existence of a unique weak solution in this sense follows. It can be checked directly that this solution is given by (3.2.10). Using the Fourier definition of the H^s norm, the same inequality immediately gives the regularity assertion:

$$\|u\|^2_{H^{s+2}} = \sum_{m \in \mathbb{Z}^n \backslash \{0\}} \frac{(1+\|m\|^2)^{s+2} |f(m)|^2}{\|m\|^4} \leq 2^2 \|f\|^2_{H^2},$$

as claimed. □

Remark 3.2.5 *The significance of the condition* $\hat{f}(0) = 0$ *for weak solutions is this: if* $u \in C^2_{per}$ *is a weak solution of* (3.2.9) *and* $\hat{f}(0) = 0$ *then* u *is in fact a classical solution (i.e. it satisfies* (3.2.9) *everywhere).*

For the case of a domain with boundary, as in theorem 3.2.3, to get regularity right up to the boundary it is necessary to assume that the boundary itself is smooth: in this case the interior regularity estimate for the weak solution of (3.2.6) can be improved to

$$\|u\|_{H^2(\Omega)} \leq C'(\|f\|_{L^2(\Omega)} + \|u\|_{L^2(\Omega)}), \qquad \text{(boundary } H^2 \text{ regularity)}.$$

3.3 Stability in Sobolev spaces

Weak solutions to elliptic boundary value problems obtained via the Lax-Milgram lemma inherit a stability (well-posedness) property in the space H^1. For example in the periodic case:

Theorem 3.3.1 *Let* $a_{jk} = a_{kj} \in C^\infty([-\pi,\pi]^n)$ *and* $c \in C^\infty([-\pi,\pi]^n)$ *be smooth periodic coefficents for the elliptic operator*

$$Pu = -\sum_{jk} \partial_j(a_{jk}\partial_k u) + cu$$

and assume (3.2.3) *and* (3.2.4) *hold for some poitive constants* m, M, c_0. *Assume* $Pu = f$ *with* $f \in L^2([-\pi, \pi]^n)$, *then there exists a number* L *such that then*

$$\|u\|_{H^1} \leq L\|f(x)\|_{L^2}.$$

If $Pu_j = f_j$ *are two such solutions then*

$$\|u_1 - u_2\|_{H^1} \leq L\|f_1 - f_2\|$$

(stability or well-posedness in H^1*).*

Proof This can be proved directly by integration by parts. □

Alternatively, this type of result is an immediate and general consequence of coercivity and the Lax-Milgram formulation. Indeed, assume that $B(u_j, v) = L_j(v)$ for $j = 1, 2$ with the bilinear form B continuous and coercive as in the Lax-Milgram lemma with coercivity constant γ, and with L_1, L_2 bounded linear functionals. Then subtracting the two equations, and choosing as test function $v = u_1 - u_2$, we deduce that

$$\gamma\|u_1 - u_2\|^2 \leq B(u_1 - u_2, u_1 - u_2) = |(L_1 - L_2)(u_1 - u_2)\,| \leq \|L_1 - L_2\|\|u_1 - u_2\|\,.$$

Here the norm on linear functionals $L : X \to \mathbb{R}$ on a Hilbert space X is the dual norm

$$\|L\| \;=\; \sup_{u \in X, u \neq 0} \frac{\|L_j u\|}{\|u\|}$$

This gives the general stability estimate

$$\|u_1 - u_2\| \leq \gamma^{-1}\,\|L_1 - L_2\| \tag{3.3.1}$$

for Lax-Milgram problems.

3.4 The maximum principle

In the previous two sections we devloped techniques based on the weak formulation, which involves integration by parts ("energy" methods). For this reason it was convenient to work with operators in the form (3.2.1), (3.2.7) in which the principal term is a divergence. In the present section this is no longer particularly convenient, so the divergence form for the pricipal term will be dropped, and variable coefficient operators of the form (3.4.1) and (3.4.2) will be considered. Throughout this section the coefficients $a_{jk}(x) = a_{kj}(x)$ are continuous and will be again assumed to satisfy the uniform ellipticity condition (3.2.3) for some positive constants m, M and all $\xi \in \mathbb{R}^n$.

Recall that $\Omega \subset \mathbb{R}^n$ is always open and bounded unless otherwise stated, $\overline{\Omega}$ is its closure and $\partial\Omega$ is its boundary (always assumed smooth). The proofs of the following results are all similar to the proof of the first, which is given. In all proofs we use the following fact from linear algebra. (Recall that a symmetric matrix A is non-negative if $\xi^T A\xi \geq 0\,\forall\,\xi \in \mathbb{R}^n$.)

Lemma 3.4.1 *If A, B are real symmetric non-negative matrices. Then $Tr(AB) = \sum_{jk} A_{jk}B_{jk} \geq 0$.*

Theorem 3.4.2 (Weak maximum principle I) *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $Pu = 0$ where*

$$Pu = -\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k u + \sum_{j=1}^{n} b_j\partial_j u \tag{3.4.1}$$

is an elliptic operator with continuous coefficients and (3.2.3) *holds, then*

$$\max_{x\in\overline{\Omega}} u(x) = \max_{x\in\partial\Omega} u(x)\,.$$

Proof Let $R > 0$ be chosen such that $mR > \|b\|_{L^\infty}$, and define $u^\epsilon = u + \epsilon e^{x_1 R}$ for $\epsilon > 0$. Then $Pu^\epsilon = (-a_{11}R^2 + b_1 R)\epsilon e^{x_1 R} < 0$ since $a_{11} \geq m$ everywhere inside Ω by assumption. Now for contradiction assume there exists an interior point x_* at which u^ϵ attains a maximum point. Then at this point $\partial_j u^\epsilon(x_*) = 0$ and $\partial^2_{jk} u^\epsilon(x_*) \leq 0$ (as a symmetric matrix) and hence lemma 3.4.1 implies that $Pu^\epsilon(x_*) \geq 0$, giving a contradiction. It follows that there can never be an interior maximum, i.e. $\max_{\overline{\Omega}} u^\epsilon = \max_{\partial\Omega} u^\epsilon$. Since this holds for all $\epsilon > 0$ the result follows by taking the limit $\epsilon \downarrow 0$. □

Theorem 3.4.3 (Weak maximum principle II) *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $Pu = 0$ where*

$$Pu = -\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k u + \sum_{j=1}^{n} b_j\partial_j u + cu \tag{3.4.2}$$

is an elliptic operator with continuous coefficients and (3.2.3) *holds and $c \geq 0$ everywhere, then $\max_{x\in\overline{\Omega}} u(x) \leq \max_{x\in\partial\Omega} u^+(x)$ where $u^+ = \max\{u, 0\}$ is the positive part of the function u.*

In these theorems the phrase *weak* maximum principle is in contrast to the *strong* maximum principle (proved for harmonic functions in the next section) which asserts that if a maximum is attained at an interior point the harmonic function is (locally) constant.

Corollary 3.4.4 *In the situation of theorem 3.4.3 $\max_{x\in\overline{\Omega}} |u(x)| = \max_{x\in\partial\Omega} |u(x)|$.*

Theorem 3.4.5 (Maximum principle bound for inhomogeneous problems) *Let $u \in C(\overline{\Omega}) \cap C^2(\Omega)$ satisfy $Pu = f$ with Dirichlet data $u|_{\partial\Omega} = 0$, where*

$$Pu = -\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k u + \sum_{j=1}^{n} b_j\partial_j u + cu \tag{3.4.3}$$

is an elliptic operator with continuous coefficients and (3.2.3) *holds and*

$$c(x) \geq c_0 > 0$$

everywhere, for some constant $c_0 > 0$*, and* $f \in C(\overline{\Omega})$*, then*

$$\max_{x \in \overline{\Omega}} u(x) \leq \frac{1}{c_0} \max_{x \in \overline{\Omega}} |f(x)| .$$

If $Pu_j = f_j$ *are two such solutions then* $\max |u_1 - u_2| \leq \max |f_1 - f_2|/c_0$ *(stability or well-posedness in uniform norm).*

3.5 Harmonic functions

Definition 3.5.1 *A function* $u \in C^2(\Omega)$ *which satisfies* $\Delta u(x) = 0$ *(resp.* $\Delta u(x) \geq 0$*, resp.* $\Delta u(x) \leq 0$*) for all* $x \in \Omega$*, for an open set* $\Omega \subset \mathbb{R}^n$*, is said to be harmonic (resp. subharmonic, resp. superharmonic) in* Ω*.*

Theorem 3.5.2 *Let* u *be harmonic in* $\Omega \subset \mathbb{R}^n$ *and assume* $\overline{B_R(x)} \subset \Omega$*. Then for* $0 < r \leq R$*:*

$$u(x) \;=\; \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(y)\, dy\,, \qquad \textit{(mean value property)}. \tag{3.5.1}$$

Proof This is a consequence of the Green identity

$$\int_{\rho<|w-x|<r} (v\Delta u - u\Delta v)\, dw = \int_{|w-x|=r} (v\partial_\nu u - u\partial_\nu v)\, d\Sigma - \int_{|w-x|=\rho} (v\partial_\nu u - u\partial_\nu v)\, d\Sigma\,,$$

(where $\partial_\nu = n \cdot \nabla$ just means the normal derivative on the boundary) with the choice of $v(w) = N(w - x)$, where N is the fundamental solution for Δ:

$$\begin{aligned} N(x) &= \frac{|x|^{2-n}}{(2-n)\omega_n}\,, \qquad (n > 2) \\ &= \frac{1}{2\pi} \ln |x|\,, \qquad (n = 2)\,. \end{aligned}$$

Here $\omega_n = \int_{|x|=1} d\Sigma(x) = 2\pi^{\frac{n}{2}}/\Gamma(n/2)$ is the area of the unit sphere in $\mathbb{R}^n$. Thus on $\partial B_r(x)$ we have $v = r^{2-n}/(2-n)\omega_n, n > 2$ or $v = (\ln r)/(2\pi), n = 2$ - in other words v is constant on $\partial B_r(x)$, which implies that

$$\int_{|w-x|=r} v\partial_\nu u\, d\Sigma = v(r) \int_{|w-x|\leq r} \Delta u\, dx = 0$$

by the divergence theorem, and the harmonicity of u. Together with the corresponding formula for the normal derivative, $\partial_\nu v = r^{1-n}/\omega_n$ on $\partial B_r(x)$, this implies that

$$\lim_{\rho\to 0}\int_{|w-x|=\rho} v\partial_\nu u\, d\Sigma = 0\,, \text{ and } \lim_{\rho\to 0}\int_{|w-x|=\rho} u\partial_\nu v\, d\Sigma = u(x)$$

where we have used also the continuity of u to take the latter limit:

$$\begin{aligned}\Big|u(x) - \int_{|w-x|=r} u\partial_\nu v\, d\Sigma\Big| &= \Big|\frac{1}{\omega_n r^{n-1}}\int_{|w-x|=\rho}\big(u(x)-u(w)\big)\, d\Sigma(w)\Big| \\ &\le \sup_{|w-x|=\rho}|u(w)-u(x)| \to 0\end{aligned}$$

as $\rho \to 0$. Substituting these into the Green identity above in the limit $\rho \to 0$ gives (3.6.1).

Corollary 3.5.3 *If u is a C^2 harmonic function in an open set Ω then $u \in C^\infty(\Omega)$. In fact if u is any C^2 function in Ω for which the mean value property* (3.6.1) *holds whenever $\overline{B_r(x)} \subset \Omega$ then u is a smooth harmonic function.*

Corollary 3.5.4 (Strong maximum principle for harmonic functions) *Let $\Omega \subset \mathbb{R}^n$ be a connected open set and $u \in C(\overline{\Omega})$ harmonic in Ω with $M = \sup_{x\in\overline{\Omega}} u(x) < \infty$. Then either $u(x) < M$ for all $x \in \Omega$ or $u(x) = M$ for all $x \in \Omega$. (In words, a harmonic function cannot have an interior maximum unless it is constant on connected components).*

Corollary 3.5.5 *Let $\Omega \subset \mathbb{R}^n$ be open with bounded closure $\overline{\Omega}$, and let $u_j \in C(\overline{\Omega})$, $j = 1, 2$ be two harmonic functions in Ω with boundary values $u_j|_{\partial\Omega} = f_j$. Then*

$$\sup_{x\in\Omega}|u_1(x)-u_2(x)| \le \sup_{x\in\partial\Omega}|f_1(x)-f_2(x)|\,, \qquad \textit{(stability or well-posedness).}$$

In particular if $f_1 = f_2$ then $u_1 = u_2$.

Corollary 3.5.6 *A harmonic function $u \in C^2(\mathbb{R}^n)$ which is bounded is constant.*

Another consequence of the Green identity is the following. Let $N(x,y) = N(|x-y|)$ where N is the fundamental solution defined above.

Theorem 3.5.7 *Let u be harmonic in Ω with $\overline{\Omega}$ bounded and $u \in C^1(\overline{\Omega})$. Then*

$$u(x) = \int_{\partial\Omega}\Big[u(y)\partial_{\nu_y}N(x,y) - N(x,y)\partial_{\nu_y}u(y)\Big]\, d\Sigma(y)\,,$$

where $\partial_{\nu_y} = n\cdot\nabla_y$ just means the normal derivative in y, while ∂_ν is the normal in x. In fact the same formula holds with $N(x,y)$ replaced by any function $G(x,y)$ such that $G(x,y) - N(x,y)$ is harmonic in $y \in \Omega$ and C^1 for $y \in \overline{\Omega}$ for each $x \in \Omega$.

It is known from above that u is determined by its boundary values - to determine a harmonic function u from $u|_{\partial\Omega}$ is the *Dirichlet problem.* (The corresponding problem of determining u from its normal derivative $\partial_\nu u|_{\partial\Omega}$ is called the *Neumann problem.* To get a formula for (or understand) the solution of these problems it is sufficient to get a formula for (or understand) the correponding Green function:

Definition 3.5.8 *(i) A function $G_D = G_D(x,y)$ defined on $G_D : \Omega \times \overline{\Omega} - \{x = y\} \to \mathbb{R}$ such that (a) $G_D(x,y) - N(|x-y|)$ is harmonic in $y \in \Omega$ and continuous for $y \in \overline{\Omega}$ for each x, and (b) $G_D(x,y) = 0$ for $y \in \partial\Omega$, is a Dirichlet Green function.*

(ii) A function $G_N = G_N(x,y)$ defined on $G_N : \Omega \times \overline{\Omega} - \{x = y\} \to \mathbb{R}$ such that (a) $G_N(x,y) - N(|x-y|)$ is harmonic in $y \in \Omega$ and continuous for $y \in \overline{\Omega}$ for each x, and (b) $\partial_{\nu_y} G_N(x,y) = 0$ for $y \in \partial\Omega$, is a Neumann Green function.

Given such functions we obtain representation formulas:

$$\Delta u = 0\,, \quad u|_{\partial\Omega} = f \implies u(x) = \int_{\partial\Omega} f(y)\partial_{\nu_y} G_D(x,y)\, d\Sigma(y)\,,$$

and

$$\Delta u = 0\,, \quad \partial_\nu u|_{\partial\Omega} = g \implies u(x) = -\int_{\partial\Omega} g(y) G_N(x,y)\, d\Sigma(y)\,,$$

for $f, g \in C(\partial\Omega)$.

The function $P(x,y) = \partial_{\nu_y} G_D(x,y)$, defined for $(x,y) \in \Omega \times \partial\Omega$ is called the Poisson kernel, and is known explicitly for certain simple domains. For example, for the unit ball $\Omega = B_1$, the Poisson kernel is $P(x,y) = (1 - \|x\|^2)/\omega_n \|x-y\|^n$ and the solution of the Dirichlet problem on the unit ball is

$$u(x) = \int_{\partial B_1} f(y) \frac{(1 - \|x\|^2)}{\omega_n \|x-y\|^n}\, d\Sigma(y)\,.$$

The formula for the half-space $\Omega = \{(x,y) : x \in \mathbb{R}^n, y > 0\}$ can also be computed explicitly (exercise).

3.6 Worked problems

1. (i) Write down the fundamental solution of the operator $-\Delta$ on $\mathbb{R}^3$ and state precisely what this means.

   (ii) State and prove the mean value property for harmonic functions on $\mathbb{R}^3$.

   (iii) Let $u \in C^2(\mathbb{R}^3)$ be a harmonic function which satisfies $u(p) \geq 0$ at every point p in an open set $\Omega \subset \mathbb{R}^3$. Show that if $B(z,r) \subset B(w,R) \subset \Omega$, then

   $$u(w) \geq (\frac{r}{R})^3 u(z).$$

   Assume that $B(x,4r) \subset \Omega$. Deduce, by choosing $R = 3r$ and w, z appropriately, that

   $$\inf_{B(x,r)} u \geq 3^{-3} \sup_{B(x,r)} u.$$

[In (iii) $B(z,\rho) = \{x \in \mathbb{R}^3 : \|x - z\| < \rho\}$ is the ball of radius $\rho > 0$ centered at $z \in \mathbb{R}^3$.]

Answer (i) the distribution $\mathbf{N} \in \mathcal{S}'(\mathbb{R}^3)$ defined by the integrable function $(4\pi|\mathbf{x}|)^{-1}$ is the fundamental solution, and the precise meaning is that

$$-\int_{\mathbb{R}^3} (4\pi|\mathbf{x}|)^{-1} \Delta\phi(\mathbf{x}) d^3x = \phi(0)$$

for every Schwarz function $\phi \in \mathcal{S}(\mathbb{R}^3)$.

(ii) *Let u be harmonic in $\Omega \subset \mathbb{R}^n$ and assume $\overline{B_R(x)} \subset \Omega$. Then for $0 < r \leq R$:*

$$u(x) \;=\; \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(y)\, dy \,, \qquad \textit{(mean value property)}. \tag{3.6.1}$$

This is a consequence of the Green identity

$$\int_{\rho<|w-x|<r} (v\Delta u - u\Delta v)\, dw = \int_{|w-x|=r} (v\partial_\nu u - u\partial_\nu v)\, d\Sigma - \int_{|w-x|=\rho} (v\partial_\nu u - u\partial_\nu v)\, d\Sigma\,,$$

(where $\partial_\nu = n \cdot \nabla$ just means the normal derivative on the boundary) with the choice of $v(w) = \mathbf{N}(w - x)$, where $\mathbf{N}$ is as in (i). On $\partial B_r(x)$ we have $v = \frac{1}{4\pi r}$ - in particular v is constant on the sphere $\partial B_r(x)$, which implies that

$$\int_{|w-x|=r} v\partial_\nu u\, d\Sigma = v(r) \int_{|w-x|\leq r} \Delta u\, dx = 0$$

by the divergence theorem, and the harmonicity of u. Together with the corresponding formula for the normal derivative, $\partial_\nu v = -\frac{1}{4\pi r^2}$ on $\partial B_r(x)$, we have:

$$\lim_{\rho\to 0} \int_{|w-x|=\rho} v\partial_\nu u\, d\Sigma \;=0\,, \text{ and } \lim_{\rho\to 0} \int_{|w-x|=\rho} u\partial_\nu v\, d\Sigma \;= -u(x)$$

(having used also the continuity of u to take the latter limit.) Substituting these into the Green identity above in the limit $\rho \to 0$ gives (3.6.1).

(iii) To start with integrate (3.6.1) with respect to r to obtain:

$$u(x) \;=\; \frac{1}{|B_r|} \int_{B_r(x)} u(y)\, dy\,. \tag{3.6.2}$$

For the first bit observe that non-negativity of u implies that $\int_{B(w,R)} u \geq \int_{B(z,r)} u$ and then apply (3.6.2) to get:

$$|B_R|u(w) = \int_{B(w,R)} u \geq \int_{B(z,r)} u = |B_r|u(z)\,.$$

Dividing by $\frac{4\pi R^3}{3} = |B_R|$ gives $u(w) \geq (\frac{r}{R})^3 u(z)$. For the second part, consider any two points w, z in the ball $B(x,r)$. Then $\|w - z\| < 2r$, and therefore $B(z,r) \subset B(w,3r) \subset \Omega$ by the triangle inequality. It follows that $u(w) \geq 3^{-3}u(z)$ and since w, z are arbitrary in $B(x,r)$ that $\inf_{B(x,r)} u \geq 3^{-3} \sup_{B(x,r)} u$. (The result is called a Harnack inequality.)

2. In this question Ω is a bounded open set in $\mathbb{R}^n$ with smooth boundary, and ν is the outward pointing unit normal vector and $\partial_\nu = \nu \cdot \nabla$.

[a] (i) Let $u \in C^4(\overline{\Omega})$[4] solve

$$\Delta^2 u = f \ \text{ in } \ \Omega\,,$$
$$u = \partial_\nu u = 0 \ \text{ on } \ \partial\Omega\,,$$

for some continuous function f. Show that if $v \in C^4(\overline{\Omega})$ also satisfies $v = \partial_\nu v = 0$ on $\partial\Omega$ then

$$\int_\Omega \Delta u\, \Delta v\, dx = \int_\Omega f\, v\, dx\,.$$

Use this to formulate a notion of weak solution to the above boundary value problem in the space $H^2_\partial(\Omega) \subset H^2(\Omega)$ which is formed by taking the closure of $C_0^\infty(\Omega)$ in the Sobolev space:

$$H^2(\Omega) = \{u \in L^2(\Omega) :\ \|u\|^2_{H^2} = \sum_{\alpha:|\alpha|\le 2} \|\partial^\alpha u\|^2_{L^2} < \infty\}\,.$$

[a](ii) State the Lax-Milgram lemma. Use it to prove that there exists a unique function u in the space $H^2_\partial(\Omega)$ which is a weak solution of the boundary value problem above for $f \in L^2(\Omega)$.

[*Hint: Use regularity of the solution of the Dirichlet problem for the Poisson equation.*]

[b] Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary. Let $u \in H^1(\Omega)$ and denote

$$\bar{u} = \int_\Omega u\, d^n x \Big/ \int_\Omega d^n x\,.$$

The following Poincaré-type inequality is known to hold

$$\|u - \bar{u}\|_{L^2} \le C\|\nabla u\|_{L^2}\,,$$

where C only depends on Ω. Use the Lax-Milgram lemma and this Poincaré-type inequality to prove that the Neumann problem

$$\Delta u = f \ \text{ in } \ \Omega\,, \qquad \partial_\nu u = 0 \ \text{ on } \ \partial\Omega\,,$$

has a unique weak solution in the space

$$H^1_-(\Omega) = H^1(\Omega) \cap \{u : \Omega \to \mathbb{R}\,;\, \bar{u} = 0\}\,,$$

for arbitrary $f \in L^2$ such that $\bar{f} = 0$. Show also that if this weak solution has regularity $u \in C^2(\overline{\Omega})$ then it is a classical solution of the Neumann problem if $\bar{f} = 0$.

Show also that if there exists a classical solution $u \in C^2(\overline{\Omega})$ to this Neumann problem then necessarily $\bar{f} = 0$.

Answer [a](i) For u, v as described the Green identity gives:

$$\int_\Omega v f\, dx = \int_\Omega v\Delta^2 u\, dx = -\int_\Omega \nabla\Delta u \cdot \nabla v\, dx + \int_{\partial\Omega} v\nabla\Delta u \cdot \nu\, dS$$
$$= \int_\Omega \Delta u \cdot \Delta v\, dx - \int_{\partial\Omega} \Delta u \nabla v \cdot \nu\, dS\,,$$

[4] This means all partial derivatives up to order 4 exist inside the open set Ω, and they have continuous extensions to the closure $\overline{\Omega}$.

which gives the result. Define the bilinear form

$$B : H^2_\partial(\Omega) \times H^2_\partial(\Omega) \to \mathbb{R}$$

by $B[u,v] := \int_\Omega \Delta u \Delta v \, dx$. Then call a weak solution of the problem a function $u \in H^2_\partial(\Omega)$ such that $B[u,v] = \int_\Omega vf \, dx$ for all functions $v \in H^2_\partial$.

[a](ii) We assume for this section H is a real Hilbert space with norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. We let $\langle \cdot,\cdot \rangle$ denote the pairing of H with its dual space.
Lax-Milgram Lemma: Assume that

$$B : H \times H \to \mathbb{R}$$

is a bilinear mapping, for which there exists constants $\alpha, \beta > 0$ such that

$$|B[u,v]| \leq \alpha \|u\| \|v\| \ (u, v \in H)$$

and

$$\beta \|u\|^2 \leq B[u,u] \ (u \in H).$$

Finally, let $f : H \to \mathbb{R}$ be a bounded linear functional on H. Then there exists a unique element $u \in H$ such that

$$B[u,v] = \langle f, v \rangle$$

for all $v \in H$.

To apply this lemma to [a](i) first notice that $|B[u,v]| \leq \|u\|_{H^2_\partial(\Omega)} \|v\|_{H^2_\partial(\Omega)}$ trivially, so it is a matter to check the second (coercivity) condition. Considering the hint, regularity of the Dirichlet problem for the Poisson equation

$$\Delta u = f \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

with $f \in L^2(\Omega)$ asserts that the unique weak solution $u \in H^1_0(\Omega)$ is actually in L^2 and verifies:

$$\|u\|^2_{H^2(\Omega)} \leq K \|f\|^2_{L^2(\Omega)} = K \|\Delta u\|^2_{L^2(\Omega)}.$$

Apply this to our problem: clearly any $u \in H^2_\partial$ also lies in H^1_0, and so:

$$\frac{1}{K} \|u\|^2_{H^2_\partial(\Omega)} \leq B[u,u].$$

Therefore by Lax-Milgram there exists a unique $u \in H^2_\partial(\Omega)$ such that

$$B[u,v] = \int_\Omega fv \, dx \tag{3.6.3}$$

for all $v \in H^2_\partial(\Omega)$ i.e. u is a weak solution.

[b] Define

$$B : H^1_-(\Omega) \times H^1_-(\Omega) \to \mathbb{R}$$

by $B[u,v] := \int_\Omega \nabla u \nabla v \, dx$.

As in a) $|B[u,v]| \leq \|u\|_{H^1(\Omega)} \|v\|_{H^1(\Omega)}$. Moreover, by the Poincare inequality with $\bar{u} = 0$, we have for $u \in H^1_-(\Omega)$:

$$\|u\|^2_{H^1(\Omega)} = \|u\|^2_{L^2(\Omega)} + \|\nabla u\|^2_{L^2(\Omega)} \leq (C^2+1) B[u,u].$$

Also, since $\bar{f} = 0$:

$$\Big|\int f u dx\Big| \; = \; \Big|\int f(u-\bar{u}) dx\Big| \; \leq \; C\|f\|_{L^2} \|\nabla u\|_{L^2} \, .$$

Thus the functional $v \mapsto \int_\Omega f v \, dx$ is bounded on $H^1_-(\Omega)$. Therefore by Lax-Milgram there exists a unique $u \in H^1_-(\Omega)$ such that

$$B[u,v] = -\int_\Omega f v \, dx$$

for all $v \in H^1_-(\Omega)$ i.e. u is a weak solution.

Now if $u \in C^2(\overline{\Omega})$ then f is also continuous; choosing as test function $\phi - \bar{\phi}$ for arbitrary $\phi \in C^1(\overline{\Omega})$, and integrating by parts, we obtain

$$\int_{\partial\Omega} \partial_\nu u (\phi - \bar{\phi}) \, dS \; - \; \int_\Omega \Delta u \, (\phi - \bar{\phi}) \, dx \; = \; - \int_\Omega f \, (\phi - \bar{\phi}) \, dx \, .$$

Now the divergence theorem gives $\int_{\partial\Omega} \partial_\nu u \, dS \; = \; \int_\Omega \Delta u \, dx$, so that the terms with $\bar{\phi}$ on the left cancel, leaving:

$$\int_{\partial\Omega} \partial_\nu u \, \phi \, dS \; - \; \int_\Omega \Delta u \, \phi \, dx \; = \; - \int_\Omega f \, (\phi - \bar{\phi}) \, dx \, .$$

If $\bar{f} = 0$ then $\int_\Omega f \, \bar{\phi} \, dx = 0$ and so we obtain

$$-\int_\Omega \Delta u \, \phi \, dx \; = \; - \int_\Omega f \, \phi \, dx \, , \quad \text{for all } \phi \in C^1_0(\Omega)$$

which implies that $\Delta u = f$ (under the assumption $\bar{f} = 0$).

For the last part we assume we have a solution classical solution u of the Neumann problem. Then we integrate the Poisson equation over Ω to obtain:

$$\int_\Omega f \, dx = \int_\Omega \Delta u \, dx = \int_{\partial\Omega} \nabla u \cdot \nu \, dx = 0$$

by the divergence theorem. Therefore $\bar{f} = 0$.

3.7 Example sheet 3

1. Recall that if $u \in C^2(\mathbb{R}^3)$ and $\Delta u \geq 0$ then u is called subharmonic. State and prove a mean value property for subharmonic functions. Also state the analogous result for superharmonic functions, i.e. those C^2 functions which satisfy $\Delta u \leq 0$.

2. Let $\phi \in C(\mathbb{R}^n)$ be absolutely integrable with $\int \phi(x)dx = 1$. Assume $f \in C(\mathbb{R}^n)$ is bounded with $\sup |f(x)| \leq M < \infty$. Define $\phi_\epsilon(x) = \epsilon^{-n}\phi(x/\epsilon)$ and show

$$\phi_\epsilon * f(x) - f(x) = \int \big(f(x - \epsilon w) - f(x)\big)\phi(w)dw$$

(where the integrals are over $\mathbb{R}^n$). Now deduce the *approximation lemma*:

$$\phi_\epsilon * f(x) \to f(x) \quad \text{as } \epsilon \to 0$$

and uniformly if f is uniformly continuous. (Hint: split up the w integral into an integral over the ball $B_R = \{w : |w| < R\}$ and its complement B_R^c for large R).*Prove that if $f \in L^p(\mathbb{R}^n), 1 \leq p < \infty$ then $\lim_{\epsilon \to 0} \|\phi_\epsilon * f(x) - f(x)\|_{L^p} = 0$. (Hint: use the Minkowski inequality in Theorem 2.9.15).

*By computing the Fourier transform of the function $\gamma_{\epsilon,a}(\xi) = \exp[i\xi \cdot a - \epsilon\|\xi\|^2]$ deduce the Fourier inversion theorem from the identity $(\hat{u}, \gamma_{\epsilon,a})_{L^2} = (u, \hat{\gamma}_{\epsilon,a})_{L^2}$ of Theorem 2.2.1.

3. Starting with the mean value property for harmonic $u \in C^2(\mathbb{R}^3)$ deduce that if $\phi \in C_0^\infty(\mathbb{R}^3)$ has total integral $\int \phi(x)dx = 1$ and is radial $\phi(x) = \psi(|x|)\,, \psi \in C_0^\infty(\mathbb{R})$ then $u = \phi_\epsilon * u$ where $\phi_\epsilon(x) = \epsilon^{-3}\phi(x/\epsilon)$. Deduce that harmonic functions $u \in C^2(\mathbb{R}^3)$ are in fact C^∞. Also for $u \in C^2(\Omega)$ harmonic in an open set $\Omega \in \mathbb{R}^3$ deduce that u is smooth in the interior of Ω (interior regularity).

4. If $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ are harmonic in Ω and agree on the boundary $\partial\Omega$, show in two different ways that $u_1 = u_2$ thoroughout Ω.

5. (i) Using the Green identities show that if f_1, f_2 both lie in $\mathcal{S}(\mathbb{R}^n)$ then the corresponding Schwartzian solutions u_1, u_2 of the equation $-\Delta u + u = f$, i.e.

$$(-\Delta + 1)u_1 = f_1 \qquad (-\Delta + 1)u_2 = f_2$$

satisfy

$$(*) \qquad \int |\nabla(u_1 - u_2)|^2 + |u_1 - u_2|^2 \leq c \int |f_1 - f_2|^2$$

where the integrals are over $\mathbb{R}^n$. (This is interpreted as implying the equation $-\Delta u + u = f$ is well-posed in the H^1 norm (or "energy" norm) defined by the left hand side of (*).) Now try to improve the result so that the H^2 norm:

$$\|u\|_{H^2}^2 \equiv \sum_{|\alpha| \leq 2} \int |\partial^\alpha u|^2 dx,$$

appears on the left. (The sum is over all multi-indices of order less than or equal to 2).

(ii) Prove a maximum principle bound for u in terms of f and deduce that $\sup_{\mathbb{R}^n} |u_1 - u_2| \leq \sup_{\mathbb{R}^n} |f_1 - f_2|$.

(iii) Verify that for $f \in \mathcal{S}'(\mathbb{R}^n)$ the formula for u_f in (3.2.9) remains valid, i.e. for each $\phi \in \mathcal{S}(\mathbb{R}^n)$ there holds

$$\langle\, u_f\,,\, -\Delta\phi + \phi\,\rangle \;=\; \langle\, f\,,\, \phi\,\rangle\,.$$

6. Prove a maximum principle for solutions of $-\Delta u + V(x)u = 0$ (on a bounded domain Ω with smooth boundary $\partial\Omega$) with $V > 0$: if $u|\partial\Omega = 0$ then $u \leq 0$ in Ω. (Assume $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Hint: exclude the possibility of u having a strictly positive interior maximum).

   What does the maximum principle reduce to for one dimensional harmonic functions i.e. C^2 functions such that $u_{xx} = 0$?

7. Write down the definition of a weak H^1 solution for the equation $-\Delta u + u + V(x)u = f \in L^2(\mathbb{R}^3)$ on the domain $\mathbb{R}^3$. Assuming that V is real valued, continuous, bounded and $V(x) \geq 0$ for all x prove the existence and uniqueness of a weak solution. Formulate and prove well posedness (stability) in H^1 for this solution.

   How about the case that V is pure imaginary valued?

8. The Dirichlet problem in half-space:
   Let $H = \{(x, y) : x \in \mathbb{R}^n, y > 0\}$ be the half-space in $\mathbb{R}^{n+1}$. Consider the problem $\Delta_x u + \partial_y^2 u = 0$, where Δ_x is the Laplacian in the x variables only) and $u(x, 0) = f(x)$ with f a bounded and uniformly continuous function on $\mathbb{R}^n$. Define

$$u(x, y) = P_y * f(x) = \int_{\mathbb{R}^n} P_y(x - z) f(z) dz$$

   where $P_y(x) = \frac{2y}{c_n(|x|^2+y^2)^{\frac{n+1}{2}}}$ for $x \in \mathbb{R}^n$ and $y > 0$. Show that for an appropriate choice of c_n the function u is harmonic on H and is equal to f for $y = 0$. This is the *Poisson kernel* for half-space.
   (Hint: first differentiate carefully under the integral sign; then note that $P_y(x) = y^{-n} P_1(\frac{x}{y})$ where $P_1(x) = \frac{2}{c_n(1+|x|^2)^{\frac{n+1}{2}}}$, i.e. an approximation to the identity) and use the approximation lemma to obtain the boundary data).

   (ii) Assume that $n = 1$ and $f \in \mathcal{S}(\mathbb{R})$. Take the Fourier transform in the x variables to prove the same result.

9. Formulate and prove a maximum principle for a 2nd order elliptic equation $Pu = f$ in the case of periodic boundary conditions. Take $Pu = -\sum_{jk=1}^n a_{jk}\partial_{jk}^2 u + \sum_{j=1}^n b_j \partial_j u + cu$ with $a_{jk} = a_{kj}, b_j, c$ and f all continuous and 2π - periodic in each variable and assume u is a C^2 function with same periodicity. Assume uniform ellipticity (3.2.3) and $c(x) \geq c_0 > 0$ for all x. Formulate and prove well-posedness for $Pu = f$ in the uniform norm.

10. Formulate a notion of weak H^1 solution for the Sturm-Liouville problem $Pu = f$ on the unit interval $[0, 1]$ with inhomogeneous Neumann data: assume $Pu = -(pu')' + qu$ with $p \in C^1([0, 1])$ and $q \in C([0, 1])$ and assume there exist constants m, c_0 such that $p \geq m > 0$ and $q \geq c_0 > 0$ everywhere, and consider boundary conditions $u'(0) = \alpha$ and $u'(1) = \beta$. (Hint : start with a classical solution, multiply by a test function $v \in C^1([0, 1])$ and integrate by parts). Prove the existence and uniqueness of a weak H^1 solution for given $f \in L^2$. (*) Show that a weak solution $u \in C^2((0, 1))$ whose first derivative u' extends continuously up to the boundary of the interval, is in fact a classical solution which satisfies $u'(0) = \alpha$ and $u'(1) = \beta$.

4 Parabolic equations

In this section we consider parabolic operators of the form

$$Lu = \partial_t u + Pu$$

where

$$Pu = -\sum_{j,k=1}^{n} a_{jk}\partial_j\partial_k u + \sum_{j=1}^{n} b_j\partial_j u + cu \tag{4.0.1}$$

is an elliptic operator. Throughout this section $a_{jk} = a_{kj}, b_j, c$ are continuous functions, and

$$m\|\xi\|^2 \le \sum_{j,k=1}^{n} a_{jk}\xi_j\xi_k \le M\|\xi\|^2 \tag{4.0.2}$$

for some positive constants m, M and all x, t and ξ. The basic example is the heat, or diffusion, equation $u_t - \Delta u = 0$, which we start by solving, first for x in an interval and then in $\mathbb{R}^n$. We then show that in both situations the solutions fit into an abstract framework of what is called a *semi-group of contraction operators*. We then discuss some properties of solutions of general parabolic equations (maximum principles and regularity theory).

4.1 The heat equation on an interval

Consider the one dimensional heat equation $u_t - u_{xx} = 0$ for $x \in [0,1]$, with Dirichlet boundary conditions $u(0,t) = 0 = u(1,t)$. Introduce the Sturm-Liouville operator $Pf = -f''$, with these boundary conditions. Its eigenfunctions $\phi_m = \sqrt{2}\sin m\pi x$ constitute an orthonormal basis for $L^2([0,1])$ (with inner product $(f,g)_{L^2} = \int f(x)g(x)dx$, considering here real valued functions). The eigenvalue equation is $P\phi_m = \lambda_m\phi_m$ with $\lambda_m = (m\pi)^2$. In terms of P the equation is:

$$u_t + Pu = 0$$

and the solution with initial data

$$u(0,x) = u_0(x) = \sum (\phi_m, u_0)_{L^2}\phi_m\,,$$

is given by

$$u(x,t) = \sum e^{-t\lambda_m}(\phi_m, u_0)_{L^2}\phi_m\,. \tag{4.1.1}$$

(In this expression $\sum$ means $\sum_{m=1}^{\infty}$.) An appropriate Hilbert space is to solve for $u(\cdot,t) \in L^2([0,1])$ given $u_0 \in L^2$, but the presence of the factor $e^{-t\lambda_m} = e^{-tm^2\pi^2}$ means the solution is far more regular for $t > 0$ than for $t = 0$:

Theorem 4.1.1 *Let $u_0 = \sum(\phi_m, u_0)_{L^2}\phi_m$ be the Fourier sine expansion of a function $u_0 \in L^2([0,1])$. Then the series (4.1.1) defines a smooth function $u(x,t)$ for $t > 0$, which satisfies $u_t = u_{xx}$ and $\lim_{t\downarrow 0}\|u(x,t) - u_0(x)\|_{L^2} = 0$.*

Proof Term by term differentiation of the series with respect to x, t has the effect only of multiplying by powers of m. For $t > 0$ the exponential factor $e^{-t\lambda_m} = e^{-tm^2\pi^2}$ thus ensures the convergence of these term by term differentiated series, absolutely and uniformly in regions $t \geq \theta > 0$ for any positive θ. It follows that for positive t the series defines a smooth function, which can be differentiated term by term, and which can be seen to solve $u_t = u_{xx}$. To prove the final assertion in the theorem, choose for each positive ϵ, a natural number $N = N(\epsilon)$ such that $\sum_{N+1}^{\infty}(\phi_m, u_0)_{L^2}^2 < \epsilon^2/4^2$. Let $t_0 > 0$ be such that for $|t| < t_0$

$$\|\sum_1^N (e^{-t\lambda_m} - 1)(\phi_m, u_0)_{L^2}\phi_m\|_{L^2} \leq \frac{\epsilon}{2}\,.$$

(This is possible because it is just a finite sum, each term of which has limit zero). Then the triangle inequality gives (for $0 < t < t_0$):

$$\begin{aligned}\|u(x,t) - u_0(x)\|_{L^2} &\leq \|\sum_1^\infty (e^{-t\lambda_m} - 1)(\phi_m, u_0)_{L^2}\phi_m\|_{L^2} \\ &\leq \frac{\epsilon}{2} + 2 \times \|\sum_{N+1}^{\infty}(\phi_m, u_0)_{L^2}\phi_m\|_{L^2} \leq \epsilon\end{aligned}$$

which implies that $\lim_{t\downarrow 0}\|u(x,t) - u_0(x)\|_{L^2} = 0$ since ϵ is arbitrary. (In the last bound, the restriction to t positive is crucial because it ensures that $e^{-t\lambda_m} \leq 1$.) □

The *instantaneous smoothing* effect established in this theorem is an important property of parabolic pde. In the next section it will be shown to occur for the heat equation on $\mathbb{R}^n$ also.

The formula (4.1.1) also holds, suitably modified, when P is replaced by any other Sturm-Liouville operator with orthonormal basis of eigenfunctions ϕ_m. For example, for if $Pu = -u''$ on $[-\pi, \pi]$) with periodic boundary conditions: in this case $\lambda_m = m^2$ and $\phi_m = e^{imx}/\sqrt{2\pi}$ for $m \in \mathbb{Z}$.

4.2 The heat kernel

The heat equation is $u_t = \Delta u$ where Δ is the Laplacian on the spatial domain. For the case of spatial domain $\mathbb{R}^n$ the distribution defined by the function

$$K(x,t) = \begin{cases} \frac{1}{\sqrt{4\pi t^n}}\exp[-\frac{\|x\|^2}{4t}] & \text{if } t > 0, \\ 0 & \text{if } t \leq 0, \end{cases} \tag{4.2.1}$$

is the fundamental solution for the heat equation (in n space dimensions). This can be derived slightly indirectly: first using the Fourier transform (in the space variable x only) the following formula for the solution of the initial value problem

$$u_t = \Delta u\,, \qquad u(x,0) = u_0(x)\,, \qquad u_0 \in \mathcal{S}(\mathbb{R}^n)\,. \tag{4.2.2}$$

Let $K_t(x) = K(x,t)$ and let $*$ indicate convolution in the space variable only, then

$$u(x,t) = K_t * u_0(x) \tag{4.2.3}$$

defines for $t > 0$ a solution to the heat equation and by the approximation lemma (see question 2 sheet 3) $\lim_{t\to 0+} u(x,t) = u_0(x)$. Once this formula has been derived for $u_0 \in \mathcal{S}(\mathbb{R}^n)$ using the fourier transform it is straightforward to verify directly that it defines a solution for a much larger class of initial data, e.g. $u_0 \in L^p(\mathbb{R}^n)$, and the solution is in fact smooth for all positive t (*instantaneous smoothing*).

The *Duhamel principle* gives the formula for the inhomogeneous equation

$$u_t = \Delta u + F\,, \qquad u(x,0) = 0 \tag{4.2.4}$$

as $u(x,t) = \int_0^t U(x,t,s)ds$, where $U(x,t,s)$ is obtained by solving the family of homogeneous initial value problems:

$$U_t = \Delta U\,, \qquad U(x,s,s) = F(x,s)\,. \tag{4.2.5}$$

This gives the formula (with $F(x,t) = 0$ for $t < 0$)

$$u(x,t) = \int_0^t K_{t-s} * F(\cdot,s)\,ds = \int_0^t K_{t-s}(x-y)F(y,s)\,ds = K \circledast F(x,t)\,,$$

for the solution of (4.2.4), where $\circledast$ means space-time convolution.

4.3 Parabolic equations and semigroups

In this section we show that the solution formulae just obtained define semi-groups in the sense of definition 6.1.1.

Theorem 4.3.1 (Semigoup property - Dirichlet boundary conditions) *The solution operator for the heat equation given by* (4.1.1)

$$S(t) : u_0 \mapsto u(\cdot,t)$$

defines a strongly continuous one parameter semigroup *of contractions on the Hilbert space* $L^2([0,1])$.

Proof $S(t)$ is defined for $t \geq 0$ on $u \in L^2([0,1])$ by

$$S(t)\sum_m (\phi_m, u)_{L^2}\phi_m \;=\; \sum_m e^{-t\lambda_m}(\phi_m, u)_{L^2}\phi_m$$

and since $|e^{-t\lambda_m}| \leq 1$ for $t \geq 0$ and $\|u\|_{L^2}^2 = \sum_m (\phi_m, u)_{L^2}^2 < \infty$ this maps L^2 into L^2 and verifies the first two conditions in definition 6.1.1. The strong continuity condition (item 4 in definition 6.1.1) was proved in theorem 4.1.1. Finally, the fact that the $\{S(t)\}_{t\geq 0}$ are contractions on L^2 is an immediate consequence of the fact that $|e^{-t\lambda_m}| \leq 1$ for $t \geq 0$. □

To transfer this result to the heat kernel solution for whole space given by (4.2.3), note the following properties of the heat kernel:

- $K_t(x) > 0$ for all $t > 0, x \in \mathbb{R}^n$,
- $\int_{\mathbb{R}^n} K_t(x)dx = 1$ for all $t > 0$,
- $K_t(x)$ is smooth for $t > 0, x \in \mathbb{R}^n$, and for t fixed $K_t(\cdot) \in \mathcal{S}(\mathbb{R}^n)$,

the following result concerning the solution $u(\cdot, t) = S(t)u_0 = K_t * u_0$ follows from basic properties of integration (see appendix to §2 on integration):

- for $u_0 \in L^p(\mathbb{R}^n)$ the function $u(x,t)$ is smooth for $t > 0, x \in \mathbb{R}^n$ and satisfies $u_t - \Delta u = 0$,
- $\|u(\cdot,t)\|_{L^p} \leq \|u_0\|_{L^p}$ and $\lim_{t\to 0+}\|u(\cdot,t) - u_0\|_{L^p} = 0$ for $1 \leq p < \infty$.

From these and the approximation lemma (question 2 sheet 3) we can read off the theorem:

Theorem 4.3.2 (Semigroup property - $\mathbb{R}^n$) *(i) The formula $u(\cdot,t) = S(t)u_0 = K_t * u_0$ defines for $u_0 \in L^1$ a smooth solution of the heat equation for $t > 0$ which takes on the initial data in the sense that $\lim_{t\to 0+}\|u(\cdot,t) - u_0\|_{L^1} = 0$.*

(ii) The family $\{S(t)\}_{t\geq 0}$ also defines a strongly continuous semigroup of contractions on $L^p(\mathbb{R}^n)$ for $1 \leq p < \infty$.

(iii) If in addition u_0 is continuous then $u(x,t) \to u_0(x)$ as $t \to 0+$ and the convergence is uniform if u_0 is uniformly continuous.

The properties of the heat kernel listed above also imply a maximum principle for the heat equation, which says that the solution always takes values in between the minimum and maximum values taken on by the intial data:

Lemma 4.3.3 (Maximum principle - heat equation on $\mathbb{R}^n$) *Let $u = u(x,t)$ be given by* (4.2.3)*. If $a \leq u_0 \leq b$ then $a \leq u(x,t) \leq b$ for $t > 0, x \in \mathbb{R}^n$.*

Related maximum principle bounds hold for general second order parabolic equations, as will be shown in the next section.

4.4 The maximum principle

Maximum principles for parabolic equations are similar to the elliptic case, once the correct notion of boundary is understood. If $\Omega \subset \mathbb{R}^n$ is an open bounded subset with smooth boundary $\partial\Omega$ and for $T > 0$ we define $\Omega_T = \Omega \times (0, T]$ then the parabolic boundary of the space-time domain Ω_T is (by definition)

$$\partial_{par}\Omega_T = \overline{\Omega_T} - \Omega_T = \Omega \times \{t = 0\} \cup \partial\Omega \times [0, T] .$$

We consider variable coefficient parabolic operators of the form

$$Lu = \partial_t u + Pu$$

as in (4.0.1), still with the uniform ellipticity assumption (4.0.2) on P.

Theorem 4.4.1 *Let $u \in C(\overline{\Omega_T})$ have derivatives up to second order in x and first order in t which are continuous in Ω_T, and assume $Lu = 0$. Then*

- *if $c = 0$ (everywhere) then $\max_{\overline{\Omega_T}} u(x,t) = \max_{\partial_{par}\Omega_T} u(x,t)$, and*
- *if $c \geq 0$ (everywhere) then $\max_{\overline{\Omega_T}} u(x,t) \leq \max_{\partial_{par}\Omega_T} u^+(x,t)$, and* $\max_{\overline{\Omega_T}} |u(x,t)| = \max_{\partial_{par}\Omega_T} |u(x,t)|$.

where $u^+ = \max\{u, 0\}$ is the positive part of the function u.

Proof We prove the first case (when $c = 0$ everywhere). To prove the maximum principle bound, consider $u^\epsilon(x,t) = u(x,t) - \epsilon t$ which verifies, for $\epsilon > 0$, the strict inequality $Lu^\epsilon < 0$. First prove the result for u^ϵ:

$$\max_{\overline{\Omega_T}} u^\epsilon(x,t) = \max_{\partial_{par}\Omega_T} u^\epsilon(x,t)$$

Since $\partial_{par}\Omega_T \subset \overline{\Omega_T}$ the left side is automatically $\geq$ the right side. If the left side were strictly greater there would be a point (x_*, t_*) with $x_* \in \Omega$ and $0 < t_* \leq T$ at which the maximum value is attained:

$$u^\epsilon(x_*, t_*) = \max_{(x,t)\in\overline{\Omega_T}} u^\epsilon(x,t) .$$

By calculus first and second order conditions: $\partial_j u^\epsilon = 0$, $u^\epsilon_t \geq 0$ and $\partial^2_{ij} u^\epsilon_x \leq 0$ (as a symmetric matrix - i.e. all eigenvalues are ≤ 0). These contradict $Lu^\epsilon < 0$ at the point (x_*, t_*). Therefore

$$\max_{\overline{\Omega_T}} u^\epsilon(x,t) = \max_{\partial_{par}\Omega_T} u^\epsilon(x,t) .$$

Now let $\epsilon \downarrow 0$ and the result follows. The proof of the second case is similar. □

4.5 Regularity for parabolic equations

Consider the Cauchy problem for the parabolic equation $Lu = \partial_t u + Pu = f$, where

$$Pu = -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \sum_{j=1}^{n} b_j \partial_j u + cu \tag{4.5.1}$$

with initial data u_0. For simplicity assume that the coefficients are all smooth functions of $x, t \in \overline{\Omega_\infty}$. The weak formulation of $Lu = f$ is obtained by multiplying by a test function $v = v(x)$ and integrating by parts, leading to (where $(\,\cdot\,)$ means the L^2 inner product defined by integration over $x \in \Omega$):

$$(u_t\,,\, v\,) + B(u,v) = (f,v)\,, \tag{4.5.2}$$

$$B(u,v) = \int \big(\sum_{jk} a_{jk}\,\partial_j u \partial_k v + \sum b_j \partial_j uv\, + cuv\big)\,dx\,.$$

To give a completely precise formulation it is necessary to define in which sense the time derivative u_t exists. To do this in a natural and general way requires the introduction of Sobolev spaces H^s for negative s - see §5.9 and §7.1.1-§7.1.2 in the book of Evans. However stronger assumptions on the initial data and inhomogeneous term are made a simpler statement is possible. (In the following statement $u(t)$ means the almost everywhere defined function of t taking values in a space of functions of x.)

Theorem 4.5.1 *For $u_0 \in H_0^1(\Omega)$ and $f \in L^2(\Omega_T)$ there exists*

$$u \in L^2([0,T]; H^2(\Omega) \cap L^\infty([0,T]; H_0^1(\Omega))$$

with time derivative $u_t \in L^2(\Omega_T)$ which satisfies (4.5.2) for all $v \in H_0^1(\Omega)$ and almost every $t \in [0,T]$ and $\lim_{t\to 0+} \|u(t) - u_0\|_{L^2} = 0$. Furthermore it is unique and has the parabolic regularity *property:*

$$\int_0^T (\|u(t)\|^2_{H^2(\Omega)} + \|u_t\|_{L^2(\Omega)}\,)\,dt + \text{ess}\sup_{0\le t\le T} \|u(t)\|^2_{H_0^1(\Omega)} \le C(\|f\|_{L^2(\Omega_T)} + \|u_0\|_{H_0^1(\Omega)})\,. \tag{4.5.3}$$

The time derivative is here to be understood in a weak/distributional sense as discussed in the sections of Evans' book just referenced, and the proof of the regularity (4.5.3) is in §7.1.3 of the same book. In the following result we will just verify that the bound holds for smooth solutions of the inhomogeneous heat equation on a periodic interval:

Theorem 4.5.2 *The Cauchy problem*

$$u_t - u_{xx} = f\,, \quad u(x,0) = u_0(x)$$

where $f = f(x,t)$ is a smooth function which is 2π-periodic in x, and the initial value u_0 is also smooth and 2π-periodic, admits a smooth solution for $t > 0$, 2π-periodic in x, which verifies the parabolic regularity estimate:

$$\int_0^T (\, \|u_t(t)\|_{L^2}^2 + \|u(t)\|_{H^2}^2 \,)\, dt \le C\, (\, \|u_0\|_{H^1}^2 + \int_0^T \int_{-\pi}^{\pi} |f(x,t)|^2 \, dx dt \,)\,.$$

Here the norms inside the time integral are the Sobolev norms on 2π-periodic functions of x taken at fixed time.

Proof To prove existence, search for a solution in Fourier form, $u = \sum \hat{u}(m,t)e^{imx}$ and obtain the ODE

$$\partial_t \hat{u}(m,t) + m^2 \hat{u}(m,t) = \hat{f}(m,t)\,, \quad \hat{u}(m,0) = \hat{u}_0(m)$$

which has solution

$$\hat{u}(m,t) = e^{-m^2 t}\hat{u}_0(m) + \int_0^t e^{-m^2(t-s)} \hat{f}(m,s)\, ds\,.$$

Now by properties of Fourier series, $\hat{u}_0(m)$ is a rapidly decreasing sequence, and the same is true for $\hat{f}(m,t)$ locally uniformly in time, since

$$\max_{0\le t\le T} m^j |\hat{f}(m,t)| \le \frac{1}{2\pi} \int_{-\pi}^{\pi} \max_{0\le t\le T} |\partial_x^j f(x,t)|\, dx\,.$$

Now, estimating $\hat{u}(m,t)$ for $0 \le t \le T$ simply as

$$|\hat{u}(m,t)| \le |\hat{u}_0(m)| + |T| \max_{0\le t\le T} |\hat{f}(m,t)|\,,$$

we see that $\hat{u}(m,t)$ is a rapidly decreasing sequence since $\hat{u}_0(m)$ and $\hat{f}(m,t)$ are. Differentiation in time just gives factors of m^2, and so $\partial_t^j \hat{u}(m,t)$ is also rapidly decreasing for each $j \in \mathbb{N}$. Therefore $u = \sum \hat{u}(m,t)e^{imx}$ defines a smooth function for positive time, and it verifies the equation (by differentiation through the sum, since this is allowed by rapidly decreasing property just established.)

To obtain the estimate, we switch to energy methods: multiply the equation by u_t and integrate. This leads to

$$\int_0^T \int_{-\pi}^{\pi} u_t^2 \, dx dt + \int_{-\pi}^{\pi} u_x^2 \, dx \big|_{t=T} = \int_{-\pi}^{\pi} u_x^2 \, dx \big|_{t=0} + \int_0^T \int_{-\pi}^{\pi} f u_t \, dx dt\,.$$

Using the Hölder inequality on the final term, this gives an estimate

$$\int_0^T \|u_t(t)\|_{L^2}^2 \, dt + \max_{0\le t\le T} \|u(t)\|_{H^1}^2 \;\le\; C \Big(\|u_0\|_{H^1}^2 + \int_0^T \int_{-\pi}^{\pi} |f(x,t)|^2 \, dx dt \Big)\,.$$

(Here and below $C > 0$ is just a positive constant whose precise value is not important). To obtain the full parabolic regularity estimate from this, it is only necessary to use the equation itself to estimate

$$\int_0^T \|u_{xx}(t)\|_{L^2}^2 \, dt \;\le\; C\Big(\int_0^T \|u_t(t)\|_{L^2}^2 \, dt \;+\; \int_0^T \|f(x,t)\|_{L^2}^2 \, dt\Big) \,,$$

and combining this with the previous bound completes the proof. □

The parabolic regularity estimate in this theorem can alternatively be derived from the Fourier form of the solution (exercise).

5 Hyperbolic equations

A second order equation of the form

$$u_{tt} + \sum_j \alpha_j \partial_t \partial_j u + Pu = 0$$

with P as in (4.0.1) (with coefficients potentially depending upon t and x), is strictly hyperbolic if the principal symbol

$$\sigma(\tau, \xi; t, x) = -\tau^2 - (\alpha \cdot \xi)\tau + \sum_{jk} a_{jk} \xi_j \xi_k$$

considered as a polynomial in τ has two distinct real roots $\tau = \tau_\pm(\xi; t, x)$ for all nonzero ξ. We will mostly study the wave equation

$$u_{tt} - \Delta u = 0 \,, \tag{5.0.4}$$

starting with some representations of the solution for the wave equation. In this section we write $u = u(t,x)$, rather than $u(x,t)$, for functions of space and time to fit in with the most common convention for the wave equation.

5.1 The one dimensional wave equation: general solution

Introducing characteristic coordinates $X_\pm = x \pm t$, the wave equation takes the form $\partial^2_{X_+ X_-} u = 0$, which has general classical solution $F(X_-) + G(X_+)$, for arbitrary C^2 functions F, G (by calculus). Therefore, the general C^2 solution of $u_{tt} - u_{xx} = 0$ is

$$u(t,x) = F(x-t) + G(x+t)$$

for arbitrary C^2 functions F, G. (This can be proved by changing to the characteristic coordinates $X_\pm = x \pm t$, in terms of which the wave equation is $\frac{\partial^2 u}{\partial X_+ \partial X_-} = 0$.

From this can be derived the solution at time $t > 0$ of the inhomogeneous initial value problem:

$$u_{tt} - u_{xx} = f \tag{5.1.1}$$

with initial data

$$u(0, x) = u_0(x) , \qquad u_t(0, x) = u_1(x) . \tag{5.1.2}$$

$$u(t, x) = \frac{1}{2}\big(u_0(x - t) + u_0(x + t)\big) + \frac{1}{2}\int_{x-t}^{x+t} u_1(y)\, dy + \frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(s, y)\, dyds . \tag{5.1.3}$$

Notice that there is again a "Duhamel principle" for the effect of the inhomogeneous term since

$$\frac{1}{2}\int_0^t \int_{x-t+s}^{x+t-s} f(s, y)\, dyds \;=\; \int_0^t U(t, s, x)ds$$

where $U(t, s, x)$ is the solution of the *homogeneous* problem with data $U(s, s, x) = 0$ and $\partial_t U(s, s, x) = f(s, x)$ specified at $t = s$.

Theorem 5.1.1 *Assuming that* $(u_0, u_1) \in C^2(\mathbb{R}) \times C^1(\mathbb{R})$ *and that* $f \in C^1(\mathbb{R} \times \mathbb{R})$ *the formula* (5.1.2) *defines a* $C^2(\mathbb{R} \times \mathbb{R})$ *solution of the wave equation, and furthermore for each fixed time* t*, the mapping*

$$\begin{aligned} C^r \times C^{r-1} &\to C^r \times C^{r-1} \\ (u_0(\cdot), u_1(\cdot)) \;&\mapsto (u(t, \cdot), u_t(t, \cdot)) \end{aligned}$$

is continuous for each integer $r \geq 2$ *. (Well-posedness in* $C^r \times C^{r-1}$*.)*

The final property stated in the theorem does not hold in more than one space dimension (question 7). This is the reason Sobolev spaces are more appropriate for the higher dimensional wave equation.

5.2 The one dimensional wave equation on an interval

Next consider the problem $x \in [0, 1]$ with Dirichlet boundary conditions $u(t, 0) = 0 = u(t, 1)$. Introduce the Sturm-Liouville operator $Pf = -f''$, with these boundary conditions as in §4.1, its eigenfunctions being $\phi_m = \sqrt{2}\sin m\pi x$ with eigenvalues $\lambda_m = (m\pi)^2$. In terms of P the wave equation is:

$$u_{tt} + Pu = 0$$

and the solution with initial data

$$u(0, x) = u_0(x) = \sum \widehat{u_0}(m)\phi_m , \qquad u_t(0, x) = u_1(x) = \sum \widehat{u_1}(m)\phi_m ,$$

is given by

$$u(t,x) = \sum_{m=1}^{\infty} \cos(t\sqrt{\lambda_m})\widehat{u_0}(m)\phi_m + \frac{\sin(t\sqrt{\lambda_m})}{\sqrt{\lambda_m}}\widehat{u_1}(m)\phi_m$$

with an analogous formula for u_t. Recall the definition of the Hilbert space $H_0^1((0,1))$ as the closure of the functions in $C_0^\infty((0,1))$[5] with respect to the norm given by $\|f\|_{H^1}^2 = \int_0^1 f^2 + f'^2\,dx$. In terms of the basis ϕ_m the definition is:

$$H_0^1((0,1)) = \{f = \sum \hat{f}_m\phi_m : \|f\|_{H^1}^2 = \sum_{m=1}^{\infty}(1+m^2\pi^2)|\hat{f}_m|^2 < \infty\}\,.$$

(In all these expressions $\sum$ means $\sum_{m=1}^{\infty}$.) As equivalent norm we can take $\sum \lambda_m|\hat{f}_m|^2$. An appropriate Hilbert space for the wave equation with these boundary conditions is to solve for $(u,u_t) \in X$ where $X = H_0^1 \oplus L^2$, and precisely we will take the following:

$$X = \{(f,g) = (\sum \hat{f}_m\phi_m, \sum \hat{g}_m\phi_m) : \|(f,g)\|_X^2 = \sum(\lambda_m|\hat{f}_m|^2 + |\hat{g}_m|^2) < \infty\}\,.$$

Now the effect of the evolution on the coefficients $\widehat{u}(m,t)$ and $\widehat{u_t}(m,t)$ is the map

$$\begin{pmatrix}\widehat{u}(m,t)\\ \widehat{u_t}(m,t)\end{pmatrix} \mapsto \begin{pmatrix}\cos(t\sqrt{\lambda_m}) & \frac{\sin(t\sqrt{\lambda_m})}{\sqrt{\lambda_m}}\\ -\sqrt{\lambda_m}\sin(t\sqrt{\lambda_m}) & \cos(t\sqrt{\lambda_m})\end{pmatrix}\begin{pmatrix}\widehat{u}(m,0)\\ \widehat{u_t}(m,0)\end{pmatrix} \tag{5.2.1}$$

Theorem 5.2.1 *The solution operator for the wave equation*

$$S(t) : \begin{pmatrix}u_0\\ u_1\end{pmatrix} \mapsto \begin{pmatrix}u(t,\cdot)\\ u_t(t,\cdot)\end{pmatrix}$$

defined by (5.2.1) *defines a strongly continuous* group *of unitary operators on the Hilbert space X, as in definition 6.3.1.*

5.3 The wave equation on $\mathbb{R}^n$

To solve the wave equation on $\mathbb{R}^n$ take the Fourier transform in the space variables to show that the solution is given by

$$u(t,x) = (2\pi)^{-n}\int \exp^{i\xi\cdot x}(\cos(t\|\xi\|)\widehat{u_0}(\xi) + \frac{\sin(t\|\xi\|)}{\|\xi\|}\widehat{u_1}(\xi))d\xi \tag{5.3.1}$$

for initial values $u(0,x) = u_0(x), u_t(0,x) = u_1(x)$ in $\mathcal{S}(\mathbb{R}^n)$. The Kirchhoff formula arises from some further manipulations with the fourier transform in the case $n=3$ and $u_0 = 0$ and gives the following formula

$$u(t,x) = \frac{1}{4\pi t}\int_{y:\|y-x\|=t} u_1(y)\,d\Sigma(y) \tag{5.3.2}$$

[5]i.e. smooth functions which are zero outside of a closed set $[a,b] \subset (0,1)$

for the solution at time $t > 0$ of $u_{tt} - \Delta u = 0$ with initial data $(u, u_t) = (0, u_1)$. The solution for the inhomogeneous initial value problem with general Schwartz initial data u_0, u_1 can then be derived from the Duhamel principle, which takes the same form as in one space dimension (as explained in §5.1).

5.4 The energy identity and finite propagation speed

Lemma 5.4.1 (Energy identity) *If u is a C^2 solution of the wave equation* (5.0.4)*, then*

$$\partial_t\Big(\frac{u_t^2 + |\nabla u|^2}{2}\Big) + \partial_i\big(-u_t\partial_i u\big) = 0$$

where $\partial_i = \frac{\partial}{\partial x^i}$.

From this and the divergence theorem some important properties follow:

Theorem 5.4.2 (Finite speed of propagation) *If $u \in C^2$ solves the wave equation* (5.0.4)*, and if $u(0,x)$ and $u_t(0,x)$ both vanish for $\|x - x_0\| < R$, then $u(t,x)$ vanishes for $\|x - x_0\| < R - |t|$ if $|t| < R$.*

Proof Notice that the energy identity can be written $\mathrm{div}_{t,x}(e,p) = 0$, where

$$(e,p) = \big(\frac{u_t^2 + |\nabla u|^2}{2}, -u_t\partial_1 u, \cdots - u_t\partial_n u\big) \in \mathbb{R}^{1+n}.$$

Let $t_0 > 0$ and consider the backwards light cone with vertex (t_0, x_0), i.e. the set

$$\{(t,x) \in \mathbb{R}^{1+3} : t \le t_0, \|x - x_0\| \le t_0 - t\}.$$

The outwards normal to this at (t,x) is $\nu = \frac{1}{\sqrt{2}}(1, \frac{x - x_0}{\|x - x_0\|}) \in \mathbb{R}^{1+n}$, which satisfies $\nu \cdot (e,p) \ge 0$ by the Cauchy-Schwarz inequality. Integrating the energy identity over the region formed by intersecting the backwards light cone with the slab $\{(t,x) \in \mathbb{R}^{1+3} : 0 \le t \le t_1\}$, and using the divergence theorem then leads to $\int_{\|x-x_0\| \le t_0 - t_1} e(t_1,x)\,dx \le \int_{\|x-x_0\|\le t_0} e(0,x)\,dx$. This implies the result by choosing $R = t_0$. □

Theorem 5.4.3 (Regularity for the wave equation) *For initial data $u(0,x) = u_0(x)$ and $u_t(0,x) = u_1(x)$ in $\mathcal{S}(\mathbb{R}^n)$, the formula* (5.3.1) *defines a smooth solution of the wave equation* (5.0.4)*, which satisfies the energy conservation law*

$$\frac{1}{2}\int_{\mathbb{R}^n} u_t(t,x)^2 + \|\nabla u(t,x)\|^2\,dx \;=\; E \;=\; constant.$$

Furthermore, at each fixed time t there holds:

$$\|(u(t,\cdot), u_t(t,\cdot))\|_{H^{s+1}\times H^s} \;\le\; C\|(u_0(\cdot), u_1(\cdot))\|_{H^{s+1}\times H^s}, \qquad C > 0 \qquad (5.4.1)$$

for each $s \in \mathbb{Z}_+$. Thus the wave equation is well-posed in the Sobolev norms $H^{s+1}\times H^s$ and regularity is preserved when measured in the Sobolev L^2 sense.

Proof The fact that (5.3.1) defines a smooth function is a consequence of the theorems on the properties of the Fourier transform and on differentiation through the integral in §2, which is allowed by the assumption that the initial data are Schwartz functions. Given this, it is straightforward to check that (5.3.1) defines a solution to the wave equation. Energy conservation follows by integrating the identity in lemma 5.4.1. Energy conservation almost gives (5.4.1) for $s = 0$. It is only necessary to bound $\|u(t,\cdot)\|_{L^2}^2$, which may be done in the following way. To start, using energy conservation, we have:

$$\Big|\frac{d}{dt}\|u\|_{L^2}^2\Big| = \big|2(u,u_t)_{L^2}\big| \leq \|u\|_{L^2}\|u_t\|_{L^2} \leq \sqrt{2E}\|u\|_{L^2}$$

This implies that $F_\epsilon(t) = (\epsilon + \|u(t,\cdot)\|_{L^2}^2)^{\frac{1}{2}}$ satisfies[6], for any positive ϵ

$$\dot{F}_\epsilon(t) \leq \sqrt{2E}$$

and hence $\|u(t,\cdot)\|_{L^2} \leq F_\epsilon(t) \leq (\epsilon + \|u(0,\cdot)\|_{L^2}^2)^{\frac{1}{2}} + \sqrt{2E}t$, for any $\epsilon > 0$. This completes the derivation of (5.4.1) for $s = 0$. The corresponding cases of (5.4.1) for $s = 1, 2\ldots$ are then derived by successively differentiating the equation, and applying the energy conservation law to the differentiated equation. □

Remark 5.4.4 *Well-posedness and preservation of regularity do not hold for the wave equation when measured in uniform norms $C^r \times C^{r-1}$, except in one space dimension, see question 7.*

Remark 5.4.5 *For initial data $(u_0, u_1) \in H^{s+1} \times H^s$ there is a distributional solution $(u(t,\cdot), u_t(t,\cdot)) \in H^{s+1} \times H^s$ at each time, which can be obtained by approximation using density of C_0^∞ in the Sobolev spaces H^s and the well-posedness estimate* (5.4.1).

6 One-parameter semigroups and groups

If A is a bounded linear operator on a Banach space its norm is

$$\|A\| = \sup_{u\in X, u\neq 0} \frac{\|Au\|}{\|u\|}, \qquad \text{(operator or uniform norm).}$$

This definition implies that if A, B are bounded linear operators on X then $\|AB\| \leq \|A\|\|B\|$.

[6]The ϵ is introduced to avoid the possibility of dividing by zero.

6.1 Definitions

Definition 6.1.1 *A one-parameter family of bounded linear operators* $\{S(t)\}_{t\geq 0}$ *on a Banach space* X *forms a semigroup if*

1. $S(0) = I$ *(the identity operator) , and*

2. $S(t+s) = S(t)S(s)$ *for all* $t, s \geq 0$ *(semi-group property).*

3. *It is called a uniformly continuous semigroup if in addition to (1) and (2):*

$$\lim_{t\to 0+} \|S(t) - I\| = 0\,, \qquad \textit{(uniform continuity).}$$

4. *It is called a strongly continuous (or* C_0*) semigroup if in addition to (1) and (2):*

$$\lim_{t\to 0+} \|S(t)u - u\| = 0\,, \forall u \in X \qquad \textit{(strong pointwise continuity).}$$

5. *If* $\|S(t)\| \leq 1$ *for all* $t \geq 0$ *the semigroup* $\{S(t)\}_{t\geq 0}$ *is called a semigroup of contractions.*

Notice that in 3 the symbol $\|\cdot\|$ *means the operator norm, while in 4 the same symbol means the norm on vectors in* X*. Also observe that uniform continuity is a stronger condition than strong continuity.*

6.2 Semigroups and their generators

For ordinary differential equations $\dot{x} = Ax$, where A is an $n \times n$ matrix, the solution can be written $x(t) = e^{tA}x(0)$ and there is a $1-1$ corespondence between the matrix A and the semigroup $S(t) = e^{tA}$ on $\mathbb{R}^n$. In this subsection[7] we discuss how this generalizes.

Uniformly continuous semigroups have a simple structure which generalizes the finite dimensional case in an obvious way - they arise as solution operators for differential equations in the Banach space X:

$$\frac{du}{dt} + Au = 0\,, \qquad \text{for } u(0) \in X \text{ given}\,. \tag{6.2.1}$$

Theorem 6.2.1 $\{S(t)\}_{t\geq 0}$ *is a uniformly continuous semgroup on* X *if and only if there exists a unique bounded linear operator* $A : X \to X$ *such that* $S(t) = e^{-tA} = \sum_{j=0}^{\infty}(-tA)^j/j!$. *This semigroup gives the solution to* (6.2.1) *in the form* $u(t) = S(t)u(0)$*, which is continuously differentiable into* X*. The operator* A *is called the infinitesimal generator of the semigroup* $\{S(t)\}_{t\geq 0}$.

[7]This subsection is for background information only

This applies to ordinary differential equations when A is a matrix. It is not very useful for partial differential equations because partial differential operators are unbounded, whereas in the foregoing theorem the infinitesimal generator was necessarily bounded. For example for the heat equation we need to take $A = -\Delta$, the laplacian defined on some appropriate Banach space of functions. Thus it is necessary to consider more general semigroups, in particular the strongly continuous semigroups. An unbounded linear operator A is a linear map from a linear subspace $D(A) \subset X$ into X (or more generally into another Banach space Y). The subspace $D(A)$ is called the domain of A. An unbounded linear operator $A : D(A) \to Y$ is said to be

- *densely defined* if $\overline{D(A)} = X$, where the overline means closure in the norm of X, and
- *closed* if the graph $\Gamma_A = \{(u, Au)|_{u \in D(A)}\} \subset X \times Y$ is closed in $X \times Y$.

A class of unbounded linear operators suitable for understanding strongly continuous semigroups is the class of *maximal monotone* operators in a Hilbert space:

Definition 6.2.2 *1. A linear operator $A : D(A) \to X$ on a Hilbert space X is monotone if $(u, Au) \geq 0$ for all $u \in D(A)$.*

2. A monotone operator $A : D(A) \to X$ is maximal monotone if, in addition, the range of $I + A$ is all of X, i.e. if:

$$\forall f \in X\, \exists u \in D(A) : (I + A)u = f\,.$$

Maximal monotone operators are automatically densely defined and closed, and there is the following generalization of theorem 6.2.1:

Theorem 6.2.3 (Hille-Yosida) *If $A : D(A) \to X$ is maximal monotone then the equation*

$$\frac{du}{dt} + Au = 0\,, \qquad \text{for } u(0) \in D(A) \subset X \text{ given}\,, \tag{6.2.2}$$

admits a unique solution $u \in C([0,\infty); D(A)) \cap C^1([0,\infty); X)$ with the property that $\|u(t)\| \leq \|u(0)\|$ for all $t \geq 0$ and $u(0) \in D(A)$. Since $D(A) \subset X$ is dense the map $D(A) \ni u(0) \to u(t) \in X$ extends to a linear map $S_A(t) : X \to X$ and by uniqueness this determines a strongly continuous semigroup of contractions $\{S_A(t)\}_{t\geq 0}$ on the Hilbert space X. Often $S_A(t)$ is written as $S_A(t) = e^{-tA}$.

Conversely, given a strongly continuous semigroup $\{S(t)\}_{t\geq 0}$ of contractions on X, there exists a unique maximal monotone operator $A : D(A) \to X$ such that $S_A(t) = S(t)$ for all $t \geq 0$. The operator A is the infinitesimal generator of $\{S(t)\}_{t\geq 0}$ in the sense that $\frac{d}{dt}S(t)u = Au$ for $u \in D(A)$ and $t \geq 0$ (interpreting the derivative as a right derivative at $t = 0$).

6.3 Unitary groups and their generators

Semigroups arise in equations which are not necessarily time reversible. For equations which are, e.g. the Schrödinger and wave equations, each time evolution operator has an inverse and the semigroup is in fact a group. In this subsection[8] We give the definitions and state the main result.

Definition 6.3.1 *A one-parameter family of unitary operators* $\{U(t)\}_{t\in\mathbb{R}}$ *on a Hilbert space* X *forms a group of unitary operators if*

1. $U(0) = I$ *(the identity operator) , and*
2. $U(t+s) = U(t)U(s)$ *for all* $t, s \in \mathbb{R}$ *(group property).*
3. *It is called a strongly continuous (or* C_0*) group of unitary operators if in addition to (1) and (2):*

$$\lim_{t\to 0} \|U(t)u - u\| = 0\,, \forall u \in X \qquad \textit{(strong pointwise continuity).}$$

A maximal monotone operator A which is symmetric (=hermitian), i.e. such that

$$(Au, v) = (u, Av) \qquad \text{for all } u, v \text{ in } D(A) \subset X \tag{6.3.1}$$

generates a one-parameter *group* of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$, often written $U(t) = e^{-itA}$, by solving the equation

$$\frac{du}{dt} + iAu = 0\,, \qquad \text{for } u(0) \in D(A) \subset X \text{ given}\,. \tag{6.3.2}$$

It is useful to introduce the adjoint operator $A*$ via the Riesz representation theorem: first of all let

$$D(A^*) = \{u \in X : \text{ the map } v \mapsto (u, Av) \text{ extends to a bounded linear functional } X \to \mathbb{C}\}$$

so that $D(A^*)$ is a linear space, and for $u \in D(A^*)$ there exists a vector w_u such that $(w_u, v) = (u, Av)$ (by Riesz representation). The map $u \to w_u$ is linear on $D(A^*)$ and so we can define an unbounded linear operator $A^* : D(A^*) \to X$ by $A^*u = w_u$, and since we started with a symmetric operator it is clear that $D(A) \subset D(A^*)$ and $A^*u = Au$ for $u \in D(A)$; the operator A^* is thus an extension of A.

Definition 6.3.2 *If* $A : D(A) \to X$ *is an unbounded linear operator which is symmetric and if* $D(A^*) = D(A)$ *then* A *is said to be self-adjoint and we write* $A = A^*$.

[8] In this subsection you only need to know definition 6.3.1. The remainder is for background information.

Theorem 6.3.3 *Maximal monotone symmetric operators are self-adjoint.*

Theorem 6.3.4 (Stone theorem) *If A is a self-adjoint operator the equation (6.3.2) has a unique solution for $u(0) \in D(A)$ which may be written $u(t) = U_A(t)u(0)$ with $\|u(t)\| = \|u(0\|$ for all $t \in \mathbb{R}$. It follows that the $U_A(t)$ extend uniquely to define unitary operators $X \to X$ and that $\{U_A(t)\}_{t\in\mathbb{R}}$ constitutes a strongly continuous group of unitary operators which are written $U_A(t) = e^{-itA}$.*

Conversely, given a strongly continuous group of unitary operators $\{U(t)\}_{t\in\mathbb{R}}$ there exists a self-adjoint operator A such that $U(t) = U_A(t) = e^{-itA}$ for all $t \in \mathbb{R}$.

6.4 Worked problems

1. Let $C^\infty_{per} = \{u \in C^\infty(\mathbb{R}) : u(x+2\pi) = u(x)\}$ be the space of smooth $2\pi-$ periodic functions of one variable.

   (i) For $f \in C^\infty_{per}$ show that there exists a unique $u = u_f \in C^\infty_{per}$ such that

   $$-\frac{\partial^2 u}{\partial x^2} + u = f.$$

   (ii) Show that $I_f[u_f + \phi] > I_f[u_f]$ for every $\phi \in C^\infty_{per}$ which is not identically zero, where $I_f : C^\infty_{per} \to \mathbb{R}$ is defined by

   $$I_f[u] = \frac{1}{2}\int_{-\pi}^{+\pi} \big((\frac{\partial u}{\partial x})^2 + u^2 - 2f(x)u\big)dx.$$

   (iii) Show that the equation

   $$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} + u = f(x),$$

   with initial data $u(0,x) = u_0(x) \in C^\infty_{per}$ has, for $t > 0$ a smooth solution $u(t,x)$ such that $u(t,\cdot) \in C^\infty_{per}$ for each fixed $t > 0$, and give a representation of this solution as a Fourier series in x. Calculate $\lim_{t\to+\infty} u(t,x)$ and comment on your answer in relation to (i).

   (iv) Show that $I_f[u(t,\cdot)] \le I_f[u(s,\cdot)]$ for $t > s > 0$, and that $I_f[u(t,\cdot)] \to I_f[u_f]$ as $t \to +\infty$.

   Answer (i) Any solution $u_f \in C^\infty_{per}$ can be represented as a Fourier series: $u_f = \sum \hat{u}_f(\alpha)e^{i\alpha x}$, as can f. Here $\alpha \in \mathbb{Z}$. The fourier coefficients are rapidly decreasing i.e. faster than any polynomial so it is permissible to differentiate through the sum, and substituting into the equation we find that the coefficients $\hat{u}_f(\alpha)$ are uniquely determined by f according to $(1+\alpha^2)\hat{u}_f(\alpha) = \hat{f}(\alpha)$, hence

   $$u_f(x) = \sum \frac{\hat{f}(\alpha)}{1+\alpha^2}e^{i\alpha x}.$$

   (Can also prove uniqueness by noting that if there were two solutions u_1, u_2 then the difference $u = u_1 - u_2$ would solve $-u_{xx} + u = 0$. Now multply by u and integrating by parts (using periodicity) - this implies that $\int u_x^2 + u^2 = 0$ which implies that $u = u_1 - u_2 = 0$.)

(ii) Calculate, using the equation satisfied by u_f and integration by parts, that

$$I_f[u_f+\phi]-I_f[u_f]=\frac{1}{2}\int_{\pi}^{\pi}(\phi_x^2+\phi^2)dx>0$$

for non-zero $\phi\in C^\infty_{per}$.

(iii) Expand the solution in terms of Fourier series and then substitute into the equation and use integrating factor to obtain that the solution is $u(t,x)=\sum\hat u(\alpha,t)e^{i\alpha x}$ where

$$\hat u(\alpha,t)=e^{-t(1+\alpha^2)}\hat u_0(\alpha)+\int_0^t e^{-(t-s)(1+\alpha^2)}\hat f(\alpha)ds.$$

Carry out the integral to deduce that

$$\hat u(\alpha,t)=\frac{\hat f(\alpha)}{1+\alpha^2}+e^{-t(1+\alpha^2)}\big(\hat u_0(\alpha)-\frac{\hat f(\alpha)}{1+\alpha^2}\big).$$

which implies that $\hat u(\alpha,t)\to\hat u_f(\alpha)=\frac{\hat f(\alpha)}{1+\alpha^2}$ as $t\to+\infty$, and further that $u(x,t)\to u_f(x)$ uniformly in x as $t\to+\infty$.

(iv) By (i) and (iii) we see that $u(x,t)=u_f(x)+\phi(x,t)$ where $\hat\phi(\alpha,t)=e^{-t(1+\alpha^2)}(\hat u_0(\alpha)-\hat u_f(\alpha))$. Now apply (ii) and use the Parseval theorem to deduce that

$$\begin{aligned}I_f[u(t,\cdot)]-I_f[u_f]&=\pi\sum(1+\alpha^2)|\hat\phi(\alpha,t)|^2\\&=\pi\sum(1+\alpha^2)e^{-2t(1+\alpha^2)}|\hat u_0(\alpha)-\hat u_f(\alpha)|^2\end{aligned}$$

which decreases to zero since $\hat u_0(\alpha)$ and $\hat u_f(\alpha)$ are rapidly decreasing.

2. For the equation $u_t-u_{xx}+u=f$, where $f=f(x,t)$ is a smooth function which is 2π-periodic in x, and the initial data $u(x,0)=u_0(x)$ are also smooth and 2π-periodic obtain the solution as a Fourier series $u=\sum\hat u(m,t)e^{imx}$ and hence verify the parabolic regularity estimate:

$$\int_0^T(\,\|u_t(t)\|^2_{L^2}+\|u(t)\|^2_{H^2}\,)\,dt\le C\,(\,\|u_0\|^2_{H^1}+\int_0^T\int_{-\pi}^{\pi}|f(x,t)|^2\,dxdt\,)\,.$$

Answer: Use the Fourier form of the solution $u(x,t)=\sum_{m\in\mathbb{Z}^n}\hat u(m,t)e^{im\cdot x}$ at each time t, and similarly for f, and the definition

$$H^s_{per}=\{u=\sum_{m\in\mathbb{Z}^n}\hat u(m)e^{im\cdot x}\in L^2:\|u\|^2_{H^s}=\sum_{m\in\mathbb{Z}^n}(1+\|m\|^2)^s|\hat u(m)|^2<\infty\}\,,$$

is for the Sobolev spaces of fixed time functions 2π-periodic in each co-ordinate x_j and for $s=0,1,2,\dots$. Writing $\omega_m=1+\|m\|^2$, and using an integrating factor the solution is given by:

$$\hat u(m,t)=e^{-\omega_m t}\hat u(m,0)+\int_0^t e^{-(t-s)\omega_m}\,\hat f(m,s)\,ds$$

in Fourier representation. The second term is a convolution, so by the Hausdorff-Young inequality $\|f*g\|^2_{L^2}\le\|f\|^2_{L^1}\|g\|^2_{L^2}$ we obtain:

$$\begin{aligned}\int_0^T|\int_0^t e^{-(t-s)\omega_m}\,\hat f(m,s)\,ds\,|^2\,dt\;&\le\;\big(\int_0^T|e^{-t\omega_m}|dt\big)^2\int_0^T|\hat f(m,t)|^2dt\\&\le\;\frac{1}{\omega_m^2}\int_0^T|\hat f(m,t)|^2dt\,.\end{aligned}$$

Here we have made use of $\int_0^T e^{-\omega_m t}\, dt = \frac{1-e^{-\omega_m T}}{\omega_m} \le \frac{1}{\omega_m}$. Using this bound, and $|a+b|^2 \le 2(a^2+b^2)$, we obtain:

$$\int_0^T \omega_m^2 |\hat{u}(m,t)|^2\, dt \le 2\Big[\int_0^T e^{-2t\omega_m}\, dt\, \omega_m^2\, |\hat{u}(m,0)|^2 + \int_0^T |\hat{f}(m,t)|^2 dt\Big]$$
$$\le 2\Big[\frac{\omega_m}{2}\, |\hat{u}(m,0)|^2 + \int_0^T |\hat{f}(m,t)|^2 dt\Big].$$

Now sum over $m \in \mathbb{Z}^n$ and use the Parseval theorem and definitions of $\|\cdot\|_{H^s}$ to obtain

$$\int_0^T \|u(t)\|_{H^2}^2\, dt \le \ const.\ \Big[\|u(0)\|_{H^1}^2 + \int_0^T |f(t)|_{L^2}^2\, dt\Big].$$

To obtain the inequality as stated it is sufficient to use the equation to obtain the same bound for $\int_0^T \|u_t(t)\|_{L^2}^2\, dt$ (with another constant).

3. (i) Define the Fourier transform $\hat{f} = \mathcal{F}(f)$ of a Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, and also of a tempered distribution $u \in \mathcal{S}'(\mathbb{R}^n)$.

(ii) From your definition compute the Fourier transform of the distribution $W_t \in \mathcal{S}'(\mathbb{R}^3)$ given by

$$W_t(\psi) =< W_t, \psi >= \frac{1}{4\pi t}\int_{\|y\|=t} \psi(y) d\Sigma(y)$$

for every Schwartz $\psi \in \mathcal{S}(\mathbb{R}^3)$. (Here $d\Sigma(y) = t^2 d\Omega(y)$ is the integration element on the sphere of radius t,) and hence deduce a formula (Kirchoff) for the solution of the initial value problem for the wave equation in three space dimensions,

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0,$$

with initial data $u(0,x) = 0$ and $\frac{\partial u}{\partial t}(0,x) = g(x), x \in \mathbb{R}^3$ where $g \in \mathcal{S}(\mathbb{R}^3)$. Explain briefly why the formula is valid for arbitrary smooth f.

(iii) Show that any C^2 solution of the initial value problem in (ii) is given by the formula derived in (ii) (uniqueness).

(iv) Show that any two solutions of the initial value problem for

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} - \Delta u = 0,$$

with identical initial data as in (ii), also agree for any $t > 0$.

Answer (i) $\hat{f}(\xi) = \int f(x) e^{-ix\cdot\xi}\, dx$, and $\langle \hat{u}, f\rangle = \langle u, \hat{f}\rangle$. This defines $u \in \mathcal{S}'(\mathbb{R}^n)$ since for any $f \in \mathcal{S}(\mathbb{R}^n)$ the Fourier transform $\hat{f} \in \mathcal{S}(\mathbb{R}^n)$ also; in fact $f \mapsto \hat{f}$ is a linear homeomorphism on $\mathcal{S}(\mathbb{R}^n)$.

(ii) Compute

$$< W_t, \hat{f} >= \frac{1}{4\pi t}\int_{\|y\|=t} \hat{f}(y) d\Sigma(y) = \frac{t}{4\pi}\int_{\mathbb{R}^n} f(\xi)\int_{\|\Omega\|=1} e^{-it\|\xi\|\cos\theta}\, d\Omega d\xi$$

Here we are writing $\Omega = (\theta, \phi)$ for the spherical polar angles for y, with the direction of ξ taken as the "e_3 axis", so that $y\cdot\xi = \|\xi\|\|y\|\cos\theta = t\|\xi\|\cos\theta$. The inner integral can be performed, after inserting $d\Omega = \sin\theta d\theta d\phi$, and equals $2\pi \times (2\sin t\|\xi\|)/(t\|\xi\|)$, so that overall:

$$< \hat{W}_t\,,\, f >=< W_t\,,\, \hat{f} >= \int_{\mathbb{R}^n} \frac{\sin t\|\xi\|}{\|\xi\|}\, f(\xi)\, d\xi\,.$$

This means $\hat{W}_t$ is the distribution determined by the function $\frac{\sin(t\|\xi\|)}{\|\xi\|}$. (This function is actually smooth and bounded by the Taylor expansion, and so determines a tempered distribution.)

But in Fourier variables the solution of the wave equation is:

$$\hat{u}(t,\xi) = \Big(\cos(t\|\xi\|)\widehat{u_0}(\xi) + \frac{\sin(t\|\xi\|)}{\|\xi\|}\widehat{u_1}(\xi)\Big)$$

for initial values $u(0,x) = u_0(x), u_t(0,x) = u_1(x)$ in $\mathcal{S}(\mathbb{R}^n)$. Comparing with the formula just derived, and applying the convolution theorem, it follows that the solution with $u_0 = 0$ and $u_1 = g$ is given at each time t by $u(t,\cdot) = W_t * g$, since then

$$\hat{u}(t,\xi) = \hat{W}_t(\xi)\hat{g}(\xi) = \frac{\sin(t\|\xi\|)}{\|\xi\|}\hat{g}(\xi)$$

(iii) Classical solutions of the wave equation obey the energy momentum conservation law

$$e_t + \nabla \cdot p = 0$$

where $e = (u_t^2 + |\nabla u|^2)/2$ and $p = -u_t \nabla u$. Integrate $e_t + \nabla \cdot p = 0$ over the part of the backward light cone with vertex (t_0, x_0), for some $t_0 > 0$, which lies in the slab between $\{t = 0\}$ and $\{t = t_1 < t_0\}$; i.e. the region

$$K_{t_0,x_0} = \{(t,x) \in \mathbb{R}^{1+3} : 0 \le t \le t_1, \|x - x_0\| \le t_0 - t\}\,.$$

Applying the divergence theorem, and noticing that if ν is the outward pointing normal on the sloping part of the boundary of this region, then $\nu \cdot (e,p) \ge 0$ by the Cauchy-Schwarz inequality, we deduce that

$$\int_{\|x-x_0\|\le t_0-t_1} e(t_1,x)\, dx \le \int_{\|x-x_0\|\le t_0} e(0,x)\, dx\,. \tag{6.4.1}$$

This implies that if the initial data are zero then the solution is zero at all later times. By time reversal symmetry an identical argument implies the same thing for negative times. Applied to the difference of two solutions this implies uniqueness (since by linearity the difference of two solutions of the wave equation also solves the wave equation), and hence that any classical C^2 solution is given by the same formula as was derived in (ii).

(iv) Do essentially the same calculation as in (iii) but using this time that

$$e_t + \nabla \cdot p \,=\, -u_t^2 \le 0$$

which gives the same conclusion 6.4.1 for positive times. (However, since time reversal symmetry no longer holds, the argument cannot now be simply reversed to obtain the analogous inequality for negative times).

6.5 Example sheet 4

1. (a) Use the change of variables $v(t,x) = e^t u(t,x)$ to obtain an "x-space" formula for the solution to the initial value problem:

$$u_t + u = \Delta u \qquad u(0,\,\cdot\,) = u_0(\cdot) \in \mathcal{S}(\mathbb{R}^n).$$

Hence show that $|u(t,x)| \le \sup_x |u_0(x)|$ and use this to deduce well-posedness in the supremum norm (for $t > 0$ and all x).

If $a \le u_0(x) \le b$ for all x what can you say about the possible values of $u(t,x)$ for $t > 0$.

(b) Use the Fourier transform in x to obtain a (Fourier space) formula for the solution of:

$$u_{tt} - 2u_t + u = \Delta u \qquad u(0,\,\cdot\,) = u_0(\cdot) \in \mathcal{S}(\mathbb{R}^n),\ u_t(0,\cdot) = u_1(\cdot) \in \mathcal{S}(\mathbb{R}^n).$$

2. Show that if $u \in C([0,\infty) \times \mathbb{R}^n) \cap C^2((0,\infty) \times \mathbb{R}^n)$ satisfies (i) the heat equation, (ii) $u(0,x) = 0$ and (iii) $|u(t,x)| \le M$ and $|\nabla u(t,x)| \le N$ for some M, N then $u \equiv 0$. (Hint: multiply heat equation by $K_{t_0 - t}(x - x_0)$ and integrate over $|x| < R, a < t < b$. Apply the divergence theorem, carefully let $R \to \infty$ and then $b \to t_0$ and $a \to 0$ to deduce $u(t_0, x_0) = 0$.)

3. Show that if $S(t)$ is a strongly continuous semigroup on a Banach space X with norm $\|\cdot\|$ then

$$\lim_{t\to 0+} \|S(t_0 + t)u - S(t_0)u\| = 0\,, \qquad \forall u \in X \text{ and } \forall t_0 > 0\,.$$

4. Let $Pu = -(pu')' + qu$, with p and q smooth, be a Sturm-Liouville operator on the unit interval $[0,1]$ and assume there exist constants m, c_0 such that $p \ge m > 0$ and $q \ge c_0 > 0$ everywhere, and consider Dirichlet boundary conditions $u(0) = 0 = u(1)$. Assume $\{\phi_n\}_{n=1}^{\infty}$ are smooth functions which constitute an orthonormal basis for $L^2([0,1])$ of eigenfunctions: $P\phi_n = \lambda_n \phi_n$. Show that there exists a number $\gamma > 0$ such that $\lambda_n \ge \gamma$ for all n. Write down the solution to the equation $\partial_t u + Pu = 0$ with initial data $u_0 \in L^2([0,1])$ and show that it defines a strongly continuous semigroup of contractions on $L^2([0,1])$, and describe the large time behaviour.

5. (i) Let $\partial_t u_j + P u_j = 0\,,\ j = 1,2$ where P is as in (4.0.1) and the functions u_j have the regularity assumed in theorem 4.4.1 and satisfy Dirichlet boundary conditions: $u_j(x,t) = 0\,\forall x \in \partial\Omega, t \ge 0$. Assuming, in addition to (4.0.2), that

$$c \ge c_0 > 0 \tag{6.5.1}$$

for some positive constant c_0 prove that for all $0 \le t \le T$:

$$\sup_{x\in\Omega} |u_1(x,t) - u_2(x,t)| \le e^{-tc_0} \sup_{x\in\Omega} |u_1(x,0) - u_2(x,0)|.$$

(ii) In the situation of part (i) with

$$Pu = -\sum_{j,k=1}^{n} \partial_j(a_{jk}\partial_k u) + \sum_{j=1}^{n} b_j \partial_j u + cu\,, \tag{6.5.2}$$

assuming in addition to (4.0.2) and (6.5.1) also that a_{jk}, b_j are C^1 and that

$$\sum_{j=1}^{n} \partial_j b_j = 0\,, \quad \text{in } \overline{\Omega_T}\,,$$

prove that for all $0 \le t \le T$:

$$\int_\Omega |u_1(x,t) - u_2(x,t)|^2\, dx \le e^{-2tc_0} \int_\Omega |u_1(x,0) - u_2(x,0)|^2\, dx.$$

6. (i) Let K_t be the heat kernel on $\mathbb{R}^n$ at time t and prove directly by integration that

$$K_t * K_s = K_{t+s}$$

for $t, s > 0$ (semi-group property). Use the Fourier transform and convolution theorem to give a second simpler proof.
(ii) Deduce that the solution operators $S(t) = K_t *$ define a strongly continuous semigroup of contractions on $L^p(\mathbb{R}^n)\, \forall p < \infty$.
(iii) Show that the solution operator $S(t) : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ for the heat initial value problem satisfies $\|S(t)\|_{L^1 \to L^\infty} \le ct^{-\frac{n}{2}}$ for positive t, or more explicitly, that the solution $u(t) = S_t u(0)$ satisfies $\|u(t)\|_{L^\infty} \le ct^{-\frac{n}{2}} \|u(0)\|_{L^1}$, or:

$$\sup_x |u(x,t)| \le ct^{-n/2} \int |u(x,0)| dx$$

for some positive number c, which should be found.
(iii) Now let $n = 4$. Deduce, by considering $v = u_t$, that if the inhomogeneous term $F \in \mathcal{S}(\mathbb{R}^4)$ is a function of x only, the solution of $u_t - \Delta u = F$ with zero initial data converges to some limit as $t \to \infty$. Try to identify the limit.

7. (i) Let $u(t,x)$ be a twice continuously differentiable solution of the wave equation on $\mathbb{R} \times \mathbb{R}^n$ for $n = 3$ which is radial, i.e. a function of $r = \|x\|$ and t. By letting $w = ru$ deduce that u is of the form

$$u(t,x) = \frac{f(r-t)}{r} + \frac{g(r+t)}{r}.$$

(ii) Show that the solution with initial data $u(0,\cdot) = 0$ and $u_t(0,\cdot) = G$, where G is radial and even function, is given by

$$u(t,r) = \frac{1}{2r} \int_{r-t}^{r+t} \rho G(\rho) d\rho.$$

(iii) Hence show that for initial data $u(0,\cdot) \in C^3(\mathbb{R}^n)$ and $u_t(0,\cdot) \in C^2(\mathbb{R}^n)$ the solution $u = u(t,x)$ need only be in $C^2(\mathbb{R} \times \mathbb{R}^n)$. Contrast this with the case of one space dimension.

8. Write down the solution of the Schrodinger equation $u_t = iu_{xx}$ with 2π-periodic boundary conditions and initial data $u(x,0) = u_0(x)$ smooth and 2π-periodic in x, and show that the solution determines a strongly continuous group of unitary operators on $L^2([-\pi,\pi])$. Do the same for Dirichlet boundary conditions i.e. $u(-\pi,t) = 0 = u(\pi,t)$ for all $t \in \mathbb{R}$.

9. (i) Write the one dimensional wave equation $u_{tt} - u_{xx} = 0$ as a first order in time evolution equation for $U = (u, u_t)$.

(ii) Use Fourier series to write down the solution with initial data $u(0, \cdot) = u_0$ and $u_t(0, \cdot) = u_1$ which are smooth 2π-periodic and have zero mean: $\hat{u}_j(0) = 0$.

(iii) Show that $\|u\|^2_{\dot{H}^1_{per}} = \sum_{m \neq 0} |m|^2 |\hat{u}(m)|^2$ defines a norm on the space of smooth 2π-periodic functions with zero mean. The corresponding complete Sobolev space is the case $s = 1$ of

$$\dot{H}^s_{per} = \{ \sum_{m \neq 0} \hat{u}(m) e^{im \cdot x} : \|u\|^2_{\dot{H}^s_{per}} = \sum_{m \neq 0} |m|^{2s} |\hat{u}(m)|^2 < \infty \} ,$$

the Hilbert space of zero mean 2π-periodic H^s functions.

(iv) Show that the solution defines a group of unitary operators in the Hilbert space

$$X = \{ U = (u, v) \ : \ u \in \dot{H}^1_{per} \text{ and } v \in L^2([-\pi, \pi]) \} .$$

(v) Explain the "unitary" part of your answer to (iv) in terms of the energy

$$E(t) = \int_{-\pi}^{\pi} (u_t^2 + u_x^2) \, dx .$$

(vi) Show that $\|U(t)\|_{\dot{H}^{s+1}_{per} \oplus \dot{H}^s_{per}} = \|(u_0, u_1)\|_{\dot{H}^{s+1}_{per} \oplus \dot{H}^s_{per}}$ (preservation of regularity).

10. (a) Deduce from the finite speed of propagation result for the wave equation (lemma 5.4.2) that a classical solution of the initial value problem, $\Box u = 0$, $u(0, t) = f$, $u_t(0, x) = g$, with $f, g \in \mathcal{D}(\mathbb{R}^n)$ given is unique.

(b) The Kirchhoff formula for solutions of the wave equation $n = 3$ for initial data $u(0, \cdot) = 0, u_t(0, \cdot) = g$ is derived using the Fourier transform when $g \in \mathcal{S}(\mathbb{R}^n)$. Show that the validity of the formula can be extended to any smooth function $g \in C^\infty(\mathbb{R}^n)$.(Hint: finite speed of propagation).

11. Write out Stone's theorem for the case of a finite dimensional Hilbert space (so that the operator is now a matrix). Prove it directly. (For the converse statement, assume differentiability of the semigroup to start with. Then convince yourself that this additional assumption is in fact redundant by considering the one dimensional case and using logarithms).

Additional questions

1. Solve the Dirichlet problem for the Laplace equation in a square $G \subset \mathbb{R}^2$

$$\Delta u = 0 \quad \text{in} \quad G = [0, a] \times [0, a] ,$$
$$u(x, 0) = f_1(x) , \qquad u(x, a) = f_2(x) , \qquad u(0, y) = f_3(y) , \qquad u(a, y) = f_4(y) .$$

[*Hint: Solve the system separately for each of the given boundary conditions, assuming that the solution is zero on the other three sides of the rectangle. Since the equation is linear, you can take the sum of these solutions as the solution of the problem.*]

Your final solution should be as follows: for the case $u(a,y) = f_4(y)$, $u(x,y) = 0$ on the rest of ∂G the solution u_{ay} is given by

$$u_{ay}(x,y) = \sum_{n=1}^{\infty} C_n \sinh\frac{n\pi}{a}x \sin\frac{n\pi}{a}y$$

where:

$$C_n = \frac{2}{a\sinh n\pi}\int_0^a f_4(y)\sin\frac{n\pi}{a}y dy \tag{6.5.3}$$

The other three solutions can be obtained through similarly: using obvious notation u_{xa} satisfying the boundary conditions for $u(x,a)$ is obtained by switching x and y, and replacing f_4 by f_2. Similarly u_{0y} and u_{x0} are found by substituting $u_{ay}(a-x,y)$ and $u_{xa}(x,a-y)$ respectively, and switching to f_3 or f_1 respectively. The complete solution is the superposition $u = u_{ay} + u_{xa} + u_{0y} + u_{x0}$ and is of the form:

$$u(x,y) = \sum_{n=1}^{\infty}\Big(C_{ayn}\sinh\frac{n\pi}{a}x\sin\frac{n\pi}{a}y + C_{xan}\sinh\frac{n\pi}{a}y\sin\frac{n\pi}{a}x$$
$$+ C_{0yn}\sinh\frac{n\pi}{a}(a-x)\sin\frac{n\pi}{a}y + C_{x0n}\sinh\frac{n\pi}{a}(a-y)\sin\frac{n\pi}{a}x\Big)$$

The coefficients are all calculated (6.5.3) up to a switch of x and y etc as just described.

2. Let $u_0 : \mathbb{R} \to \mathbb{R}$, $u_0 \in C^1(\mathbb{R})$, $u_0(x) \geq 0$ for all $x \in \mathbb{R}$. Consider the partial differential equation for $u = u(x,y)$,

$$4yu_x + 3u_y = u^2, \qquad (x,y) \in \mathbb{R}^2$$

subject to the Cauchy condition $u(x,0) = u_0(x)$.

i) Compute the solution of the Cauchy problem by the method of characteristics.

ii) Prove that the domain of definition of the solution contains

$$(x,y) \in \mathbb{R} \times \left(-\infty, \frac{3}{\sup_{x\in\mathbb{R}}(u_0(x))}\right).$$

3. Define the tempered distributions $\mathcal{S}'(\mathbb{R})$ and the Dirac distribution $\delta_a(x) = \delta(x-a)$ and prove that δ_a does define a tempered distribution $\delta_a \in \mathcal{S}'(\mathbb{R})$.

Show also that if $t \to X(t)$ is continuous then the formula $T(x,t) = \delta(x - X(t))$ defines a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^2)$.

Define the distributional derivative of a tempered distribution. Starting from this definition prove that the tempered distribution T just defined satisfies the partial differential equation

$$\partial_t T + x\partial_x T + T = 0$$

if $X(t)$ is a C^1 solution of the equation $\dot{X} = X$.

4. Calculate the fourier transform of the distribution $T_t \in \mathcal{S}'(\mathbb{R}^3)$ defined, for all $t > 0$, by:

$$\langle T_t, \chi\rangle = \frac{1}{4\pi t}\int\limits_{\|y-x\|=t} \chi(y)\, d\Sigma(y)\,, \qquad \chi \in \mathcal{S}(\mathbb{R}^3)\,. \tag{6.5.4}$$

Relate your answer to the wave equation in three space dimensions.

5. For the equation $u_t - u_{xx} + u = f$, where $f = f(x,t)$ is a smooth function which is 2π-periodic in x, and the initial data $u(x,0) = u_0(x)$ are also smooth and 2π-periodic obtain the solution as a Fourier series $u = \sum \hat{u}(m,t)e^{imx}$ and hence verify the parabolic regularity estimate:

$$\int_0^T (\, \|u_t(t)\|_{L^2}^2 + \|u(t)\|_{H^2}^2 \,)\, dt \le C\,(\, \|u_0\|_{H^1}^2 + \int_0^T \int_{-\pi}^{\pi} |f(x,t)|^2 \, dxdt \,)\,.$$

6. Write down the solution of

$$u_t = u_{xxx} \qquad u(0,x) = u_0(x) \tag{6.5.5}$$

for u_0 smooth and 2π-periodic, and show that the solution operator defines a strongly continuous group of unitary operators on $L^2([-\pi,\pi])$.

7. Prove that there exists a constant C such that for $s > n/2$

$$\max_{x \in \mathbb{R}^n} |u(x)| \le C\|u\|_{H^s} \tag{6.5.6}$$

for a smooth function which is 2π-periodic in each co-ordinate $\{x_j\}_{j=1}^n$. Deduce from the density of such smooth functions in H^s_{per} that (6.5.6) holds for all $u \in H^s_{per}$.

For the equation

$$u_{tt} - \Delta u + u = 0 \tag{6.5.7}$$

with smooth initial data $u(0,x) = u_0(x)$ and $u_t(0,x) = u_1(x)$ which is 2π-periodic in each coordinate show that the solution satisfies at each $T > 0$

$$\max_{0 \le t \le T} \|(u(t), u_t(t))\|^2_{H^{s+1} \times H^s} \le C' \|(u_0, u_1)\|^2_{H^{s+1} \times H^s}$$

for every $s = 0, 1, 2 \ldots$ and some positive constant C'.

Find a value of s (depending upon n) which ensures that if $\|(u_0, u_1)\|_{H^{s+1} \times H^s} < \infty$ then $u(t,x)$ remains bounded for all time. Do the same for an arbitrary partial derivative $\partial^\alpha u(t,x)$.

(*In this question use*

$$H^s_{per} = \{u = \sum_{m \in \mathbb{Z}^n} \hat{u}(m)e^{im\cdot x} \in L^2 : \|u\|^2_{H^s} = \sum_{m \in \mathbb{Z}^n} (1 + \|m\|^2)^s |\hat{u}(m)|^2 < \infty\}\,,$$

for the Sobolev spaces of functions 2π-periodic in each co-ordinate x_j and for $s = 0, 1, 2, \ldots$.)

8. For the equation

$$u_{tt} - u_{xx} + u = 0\,, \tag{6.5.8}$$

with Schwartz initial data $u(0,\cdot) = u_0(\cdot) \in \mathcal{S}(\mathbb{R})$ and $u_t(0,\cdot) = u_1(\cdot) \in \mathcal{S}(\mathbb{R})$ show that the solution satisfies at each $T > 0$

$$\max_{0 \le t \le T} \|(u(t), u_t(t))\|^2_{H^{s+1} \times H^s} \le C \|(u_0, u_1)\|^2_{H^{s+1} \times H^s}$$

for every $s = 0, 1, 2 \ldots$ and some constant $C = C(m, T)$. Comment on the dependence on T, distinguishing the case $m = 0$.

Prove that a C^2 solution verfies

$$\partial_t\left(\frac{u_t^2+u_x^2+u^2}{2}\right)-\partial_x\Big(u_t u_x\Big)=0$$

and deduce from this and the divergence theorem, applied on the solid backward light cone, a finite propagation speed result for (6.5.8).

9. (i) Derive, from the definition of Fourier transform on $\mathcal{S}'(\mathbb{R}^n)$, a formula for the Fourier transform of δ_0 and its derivatives. For $n=1$ find the Fourier transform of the Heaviside distribution.

   (ii) If $T\in\mathcal{S}'(\mathbb{R}^n)$ and $\phi\in\mathcal{S}(\mathbb{R}^n)$, prove that $\widehat{\phi T}=(2\pi)^{-n}\widehat{\phi}\star\widehat{T}$.

10. Write down the Kirchoff formula for the solution of the wave equation in three space dimensions with initial data $u(0,x)=0, u_t(0,x)=u_1(x)$ for $u_1\in\mathcal{S}(\mathbb{R}^3)$. What can you say about the behaviour of $u(t,x)$ for large t?

    What can you say about the behaviour of $\int|u(t,x)|^2dx$ for large t? How about for the equation (6.5.8)?

11. (i) Use the Fourier transform to obtain a representation for the solution to the initial value problem (6.5.5) for u_0 a Schwartz function. Deduce that the solution obtained satisfies the well-posedness estimate for all times t:

$$\int_{-\infty}^{+\infty}|u(t,x)-v(t,x)|^2dx\le\int_{-\infty}^{+\infty}|u(0,x)-v(0,x)|^2dx$$

    (Global well-posedness in L^2 uniformly in time). Show that the formula defines a strongly continuous group of unitary operators on $L^2(\mathbb{R})$.

12. Using the representation of the solution of the initial value problem

$$(**)\qquad\qquad \partial_t u-\Delta u=0\qquad u(0,x)=f(x),\quad f\in\mathcal{S}(\mathbb{R}^n)$$

    given by the fundamental solution show that if $\sup_{x\in\mathbb{R}^n}f(x)\le M$ then $\sup_{x\in\mathbb{R}^n}u(t,x)\le M$ for all $t>0$. Let u be a smooth solution of the initial value problem (**) which lies in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ for each fixed $t>0$. Compute, assuming you can differentiate under the integral sign, $\frac{d}{dt}\int_{\mathbb{R}^n}|u(t,x)|^2dx$ and hence prove that there is only one such solution of (**). Compute also

$$\frac{d}{dt}\int_{\mathbb{R}^n}\phi\big(u(t,x)\big)\,dx$$

    where $\phi\in C^2(\mathbb{R})$ is a positive function. For which ϕ is your answer ≤ 0?

13. (i) Starting from the Kirchoff formula in the case that $u_t(0,x)=u_1(x)$ depends only on x_1,x_2 and is independent of x_3, obtain a formula for the solution of the wave equation $u_{tt}-\Delta u=0$ in two space dimensions with initial data $u(0,x_1,x_2)=0$ and $u_t(0,x_1,x_2)=u_1(x_1,x_2)$.

    (ii) For the solution of the inhomogeneous Cauchy problem $\Box u=h$ with h a Schwartz function as obtained by the Duhamel principle determine the domain of dependence for a point (t_0,x_0) on the values of h. Comment on the difference between the cases of two and three space dimensions.

Old exam questions

1. (a) Solve the equation

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = u^2$$

together with the boundary condition on the x-axis:

$$u(x,0) = f(x) \ ,$$

where f is a smooth function. You should discuss the domain on which the solution is smooth. For which functions f can the solution be extended to give a smooth solution on the upper half plane $\{y > 0\}$?

(b) Solve the equation

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$$

together with the boundary condition on the unit circle:

$$u(x,y) = x \qquad \text{when} \qquad x^2 + y^2 = 1.$$

2. Define the Schwartz space $\mathcal{S}(\mathbb{R})$ and the corresponding space of tempered distributions $\mathcal{S}'(\mathbb{R})$.

Use the Fourier transform to give an integral formula for the solution of the equation

$$-\frac{d^2u}{dx^2} + \frac{du}{dx} + u = f \qquad (*)$$

for $f \in \mathcal{S}(\mathbb{R})$. Prove that your solution lies in $\mathcal{S}(\mathbb{R})$. Is your formula the unique solution to $(*)$ in the Schwartz space?

Deduce from this formula an integral expression for the fundamental solution of the operator $P = -\frac{d^2}{dx^2} + \frac{d}{dx} + 1$.

Let K be the function:

$$K(x) = \begin{cases} \frac{1}{\sqrt{5}} e^{-(\sqrt{5}-1)x/2} & \text{for} x \geq 0, \\ \frac{1}{\sqrt{5}} e^{(\sqrt{5}+1)x/2} & \text{for} x \leq 0. \end{cases}$$

Using the definition of distributional derivatives verify that this function is a fundamental solution for P.

3. Write down a formula for the solution $u = u(t,x)$, for $t > 0$ and $x \in \mathbb{R}^n$, of the initial value problem for the heat equation:

$$\frac{\partial u}{\partial t} - \Delta u = 0 \qquad u(0,x) = f(x),$$

for f a bounded continuous function $f : \mathbb{R}^n \to \mathbb{R}$. State (without proof) a theorem which ensures that this formula is the unique solution in some class of functions (which should be explicitly described).

By writing $u = e^t v$, or otherwise, solve the initial value problem

$$\frac{\partial v}{\partial t} + v - \Delta v = 0, \qquad v(0,x) = g(x), \tag{†}$$

for g a bounded continuous function $g : \mathbb{R}^n \to \mathbb{R}$ and give a class of functions in which your solution is the unique one.

Hence, or otherwise, prove that for all $t > 0$:

$$\sup_{x \in \mathbb{R}^n} v(t,x) \leq \sup_{x \in \mathbb{R}^n} g(x)$$

and deduce that the solutions $v_1(t,x)$ and $v_2(t,x)$ of (†) corresponding to initial values $g_1(x)$ and $g_2(x)$ satisfy, for $t > 0$,

$$\sup_{x \in \mathbb{R}^n} |v_1(t,x) - v_2(t,x)| \leq \sup_{x \in \mathbb{R}^n} |g_1(x) - g_2(x)|.$$

4. (a) Solve the equation, for a function $u(x,y)$,

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0 \tag{$*$}$$

together with the boundary condition on the x-axis:

$$u(x,0) = x.$$

Find for which real numbers a it is possible to solve ($*$) with the following boundary condition specified on the line $y = ax$:

$$u(x,ax) = x.$$

Explain your answer in terms of the notion of *characteristic hypersurface*, which should be defined.

(b) Solve the equation

$$\frac{\partial u}{\partial x} + (1+u)\frac{\partial u}{\partial y} = 0$$

with the boundary condition on the x-axis

$$u(x,0) = x,$$

in the domain $\mathcal{D} = \{(x,y) : 0 < y < (x+1)^2/4, \ -1 < x < \infty\}$. Sketch the characteristics.

5. (a) Define the convolution $f * g$ of two functions. Write down a formula for a solution $u : [0,\infty) \times \mathbb{R}^n \to \mathbb{R}$ to the initial value problem

$$\frac{\partial u}{\partial t} - \Delta u = 0$$

together with the boundary condition

$$u(0,x) = f(x)$$

for f a bounded continuous function on $\mathbb{R}^n$. Comment briefly on the uniqueness of the solution.

(b) State and prove the Duhamel principle giving the solution (for $t > 0$) to the equation

$$\frac{\partial u}{\partial t} - \Delta u = g$$

together with the boundary condition

$$u(0, x) = f(x)$$

in terms of your answer to (a).

(c) Show that if $v : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ is the solution to

$$\frac{\partial v}{\partial t} - \Delta v = G$$

together with the boundary condition

$$v(0, x) = f(x)$$

with $G(t, x) \leq g(t, x)$ for all (t, x) then $v(t, x) \leq u(t, x)$ for all $(t, x) \in (0, \infty) \times \mathbb{R}^n$.

Finally show that if in addition there exists a point (t_0, x_0) at which there is strict inequality in the assumption i.e.

$$G(t_0, x_0) < g(t_0, x_0),$$

then in fact

$$v(t, x) < u(t, x)$$

whenever $t > t_0$.

6. Define the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ and the space of tempered distributions $\mathcal{S}'(\mathbb{R}^n)$. State the Fourier inversion theorem for the Fourier transform of a Schwartz function.

Consider the initial value problem:

$$\frac{\partial^2 u}{\partial t^2} - \Delta u + u = 0 \ , \ x \in \mathbb{R}^n \ , \ 0 < t < \infty \ ,$$

$$u(0, x) = f(x) \ , \qquad \frac{\partial u}{\partial t}(0, x) = 0$$

for f in the Schwartz space $\mathcal{S}(\mathbb{R}^n)$.

Show that the solution can be written as

$$u(t, x) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{u}(t, \xi) d\xi \ ,$$

where

$$\hat{u}(t, \xi) = \cos\left(t\sqrt{1 + |\xi|^2}\right) \hat{f}(\xi)$$

and

$$\hat{f}(\xi) = (2\pi)^{-n/2} \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx.$$

State the Plancherel-Parseval theorem and hence deduce that

$$\int_{\mathbb{R}^n} |u(t, x)|^2 dx \leq \int_{\mathbb{R}^n} |f(x)|^2 dx.$$

7. (a) Define characteristic hypersurfaces and state a local existence and uniqueness theorem for a quasilinear partial differential equation with data on a non-characteristic hypersurface.

(b) Consider the initial value problem

$$3u_x + u_y = -yu, \qquad u(x,0) = f(x),$$

for a function $u : \mathbb{R}^2 \to \mathbb{R}$ with C^1 initial data f given for $y = 0$. Obtain a formula for the solution by the method of characteristics and deduce that a C^1 solution exists for all $(x, y) \in \mathbb{R}^2$.

Derive the following (*well-posedness*) property for solutions $u(x, y)$ and $v(x, y)$ corresponding to data $u(x, 0) = f(x)$ and $v(x, 0) = g(x)$ respectively:

$$\sup_x |u(x,y) - v(x,y)| \leqslant \sup_x |f(x) - g(x)| \qquad \text{for all } y.$$

(c) Consider the initial value problem

$$3u_x + u_y = u^2, \qquad u(x,0) = f(x),$$

for a function $u : \mathbb{R}^2 \to \mathbb{R}$ with C^1 initial data f given for $y = 0$. Obtain a formula for the solution by the method of characteristics and hence show that if $f(x) < 0$ for all x, then the solution exists for all $y > 0$. Show also that if there exists x_0 with $f(x_0) > 0$, then the solution does not exist for all $y > 0$.

8. (a) If f is a radial function on $\mathbb{R}^n$ (i.e. $f(x) = \phi(r)$ with $r = |x|$ for $x \in \mathbb{R}^n$) , and $n > 2$, then show that f is harmonic on $\mathbb{R}^n - \{0\}$ if and only if

$$\phi(r) = a + br^{2-n}$$

for $a, b \in \mathbb{R}$.

(b) State the mean value theorem for harmonic functions and prove it for $n > 2$.

(c) Generalise the statement and the proof of the mean value theorem to the case of a subharmonic function, i.e. a C^2 function such that $\Delta u \leqslant 0$.

9. Consider the initial value problem

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0 \tag{1}$$

to be solved for $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, subject to the initial conditions

$$u(0,x) = f(x), \quad \frac{\partial u}{\partial t}(0,x) = 0 \tag{2}$$

for f in the Schwarz space $\mathcal{S}(\mathbb{R}^n)$. Use the Fourier transform in x to obtain a representation for the solution in the form

$$u(t,x) = \int e^{ix\cdot\xi} A(t,\xi) \widehat{f}(\xi) d^n\xi \tag{3}$$

where A should be determined explicitly. Explain carefully why your formula gives a smooth solution to (1) and why it satisfies the initial conditions (2), referring to the required properties of the Fourier transform as necessary.

Next consider the case $n = 1$. Find a tempered distribution T (depending on t, x) such that (3) can be written

$$u =< T, \widehat{f} >$$

and (using the definition of Fourier transform of tempered distributions) show that the formula reduces to

$$u(t, x) = \frac{1}{2}\big[f(x - t) + f(x + t)\big].$$

State and prove the Duhamel principle relating to the solution of the n-dimensional inhomogeneous wave equation

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = h$$

to be solved for $u : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$, subject to the initial conditions

$$u(0, x) = 0, \quad \frac{\partial u}{\partial t}(0, x) = 0$$

for h a C^∞ function. State clearly assumptions used on the solvability of the homogeneous problem.

[*Hint: it may be useful to consider the Fourier transform of the tempered distribution defined by the function* $\xi \mapsto e^{i\xi\cdot a}$.]

10. (a) State and prove the Mean Value Theorem for harmonic functions.

(b) Let $u \geq 0$ be a harmonic function on an open set $\Omega \subset \mathbb{R}^n$. Let $B(x, a) = \{y \in \mathbb{R}^n : |x - y| < a\}$. For any $x \in \Omega$ and for any $r > 0$ such that $B(x, 4r) \subset \Omega$, show that

$$\sup_{\{y\in B(x,r)\}} u(y) \leq 3^n \inf_{\{y\in B(x,r)\}} u(y)\,.$$

11. (a) State and prove the Duhamel principle for the wave equation.

(b) Let $u \in C^2([0, T] \times \mathbb{R}^n)$ be a solution of

$$u_{tt} + u_t - \Delta u + u = 0$$

where Δ is taken in the variables $x \in \mathbb{R}^n$ and $u_t = \partial_t u$ etc.

Using an 'energy method', or otherwise, show that, if $u = u_t = 0$ on the set $\{t = 0, |x - x_0| \leq t_0\}$ for some $(t_0, x_0) \in [0, T] \times \mathbb{R}^n$, then u vanishes on the region $K(t, x) = \{(t, x) : 0 \leq t \leq t_0, |x - x_0| \leq t_0 - t\}$. Hence deduce uniqueness for the Cauchy problem for the above PDE with Schwartz initial data.

12. (i) Find $w : [0, \infty) \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $w(t, \cdot)$ is a Schwartz function of ξ for each t and solves

$$w_t(t, \xi) + (1 + \xi^2)w(t, \xi) = g(\xi)\,,$$
$$w(0, \xi) = w_0(\xi)\,,$$

where g and w_0 are given Schwartz functions and w_t denotes $\partial_t w$. If $\mathcal{F}$ represents the Fourier transform operator in the ξ variables only and $\mathcal{F}^{-1}$ represents its inverse, show that the solution w satisfies

$$\partial_t(\mathcal{F}^{-1})w(t,x) = \mathcal{F}^{-1}(\partial_t w)(t,x)$$

and calculate $\lim_{t\to\infty} w(t,\cdot)$ in Schwartz space.

(ii) Using the results of Part (i), or otherwise, show that there exists a solution of the initial value problem

$$u_t(t,x) - u_{xx}(t,x) + u(t,x) = f(x)$$
$$u(0,x) = u_0\,,$$

with f and u_0 given Schwartz functions, such that

$$\|u(t,\cdot) - \phi\|_{L^\infty(\mathbb{R})} \longrightarrow 0$$

as $t \to \infty$ in Schwartz space, where ϕ is the solution of

$$-\phi'' + \phi = f.$$

13. Consider the equation

$$x_2\,\frac{\partial u}{\partial x_1} - x_1\,\frac{\partial u}{\partial x_2} + a\,\frac{\partial u}{\partial x_3} = u, \qquad (*)$$

where $a \in \mathbb{R}$, to be solved for $u = u(x_1, x_2, x_3)$. State clearly what it means for a hypersurface

$$S_\phi = \big\{(x_1, x_2, x_3) :\ \phi(x_1, x_2, x_3) = 0\big\},$$

defined by a C^1 function ϕ, to be *non-characteristic for* $(*)$. Does the non-characteristic condition hold when $\phi(x_1, x_2, x_3) = x_3$?

Solve $(*)$ for $a > 0$ with initial condition $u(x_1, x_2, 0) = f(x_1, x_2)$ where $f \in C^1(\mathbb{R}^2)$. For the case $f(x_1, x_2) = x_1^2 + x_2^2$ discuss the limiting behaviour as $a \to 0_+$.

14. Define a *fundamental solution* of a linear partial differential operator P. Prove that the function

$$G(x) = \tfrac{1}{2}e^{-|x|}$$

defines a distribution which is a fundamental solution of the operator P given by

$$P\,u = -\frac{d^2u}{dx^2} + u\,.$$

Hence find a solution u_0 to the equation

$$-\frac{d^2u_0}{dx^2} + u_0 = V(x)\,,$$

where $V(x) = 0$ for $|x| > 1$ and $V(x) = 1$ for $|x| \le 1$.

Consider the functional

$$I[u] = \int_{\mathbb{R}} \left\{\frac{1}{2}\left[\left(\frac{du}{dx}\right)^2 + u^2\right] - Vu\right\} dx\,.$$

Show that $I[u_0 + \phi] > I[u_0]$ for all Schwartz functions ϕ that are not identically zero.

15. Write down a formula for the solution $u = u(t,x)$ of the n-dimensional heat equation

$$w_t(t,x) - \Delta w = 0, \qquad w(0,x) = g(x),$$

for $g : \mathbb{R}^n \to \mathbb{C}$ a given Schwartz function; here $w_t = \partial_t w$ and Δ is taken in the variables $x \in \mathbb{R}^n$. Show that

$$w(t,x) \leq \frac{\int |g(x)|\, dx}{(4\pi t)^{n/2}} .$$

Consider the equation

$$u_t - \Delta u = e^{it} f(x) , \tag{$*$}$$

where $f : \mathbb{R}^n \to \mathbb{C}$ is a given Schwartz function. Show that $(*)$ has a solution of the form

$$u(t,x) = e^{it} v(x) ,$$

where v is a Schwartz function.

Prove that the solution $u(t,x)$ of the initial value problem for $(*)$ with initial data $u(0,x) = g(x)$ satisfies

$$\lim_{t\to+\infty} \left| u(t,x) - e^{it} v(x) \right| = 0 .$$

16. Write down the solution of the three-dimensional wave equation

$$u_{tt} - \Delta u = 0 , \qquad u(0,x) = 0 , \quad u_t(0,x) = g(x) ,$$

for a Schwartz function g. Here Δ is taken in the variables $x \in \mathbb{R}^3$ and $u_t = \partial u/\partial t$ etc. State the “strong” form of Huygens principle for this solution. Using the method of descent, obtain the solution of the corresponding problem in two dimensions. State the “weak” form of Huygens principle for this solution.

Let $u \in C^2([0,T] \times \mathbb{R}^3)$ be a solution of

$$u_{tt} - \Delta u + |x|^2 u = 0 , \qquad u(0,x) = 0 , \quad u_t(0,x) = 0 . \tag{$*$}$$

Show that

$$\partial_t e + \nabla \cdot \mathbf{p} = 0 , \tag{$**$}$$

where

$$e = \tfrac{1}{2}\big({u_t}^2 + |\nabla u|^2 + |x|^2 u^2\big), \qquad \text{and} \qquad \mathbf{p} = -u_t \nabla u .$$

Hence deduce, by integration of $(**)$ over the region

$$K = \big\{(t,x) : 0 \leq t \leq t_0 - a \leq t_0, \;\; |x - x_0| \leq t_0 - t\big\}$$

or otherwise, that $(*)$ satisfies the weak Huygens principle.

17. (a) State a local existence theorem for solving first order quasi-linear partial differential equations with data specified on a smooth hypersurface.

(b) Solve the equation

$$\frac{\partial u}{\partial x} + x\frac{\partial u}{\partial y} = 0$$

with boundary condition $u(x,0) = f(x)$ where $f \in C^1(\mathbb{R})$, making clear the domain on which your solution is C^1. Comment on this domain with reference to the *non-characteristic condition* for an initial hypersurface (including a definition of this concept).

(c) Solve the equation

$$u^2 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 0$$

with boundary condition $u(x, 0) = x$ and show that your solution is C^1 on some open set containing the initial hypersurface $y = 0$. Comment on the significance of this, again with reference to the non-characteristic condition.

18. Define a *fundamental solution* of a constant-coefficient linear partial differential operator, and prove that the distribution defined by the function $N : \mathbb{R}^3 \to \mathbb{R}$

$$N(x) = (4\pi|x|)^{-1}$$

is a fundamental solution of the operator $-\Delta$ on $\mathbb{R}^3$.

State and prove the mean value property for harmonic functions on $\mathbb{R}^3$ and deduce that any two smooth solutions of

$$-\Delta u = f \ , \qquad f \in C^\infty(\mathbb{R}^3)$$

which satisfy the condition

$$\lim_{|x|\to\infty} u(x) = 0$$

are in fact equal.

19. Write down the formula for the solution $u = u(t, x)$ for $t > 0$ of the initial value problem for the n-dimensional heat equation

$$\frac{\partial u}{\partial t} - \Delta u = 0 \ ,$$

$$u(0, x) = g(x) \ ,$$

for $g : \mathbb{R}^n \to \mathbb{C}$ a given smooth bounded function.

State and prove the Duhamel principle giving the solution $v(t, x)$ for $t > 0$ to the inhomogeneous initial value problem

$$\frac{\partial v}{\partial t} - \Delta v = f \ ,$$

$$v(0, x) = g(x) \ ,$$

for $f = f(t, x)$ a given smooth bounded function.

For the case $n = 4$ and when $f = f(x)$ is a fixed Schwartz function (independent of t), find $v(t, x)$ and show that $w(x) = \lim_{t\to+\infty} v(t, x)$ is a solution of

$$-\Delta w = f \ .$$

[*Hint: you may use without proof the fact that the fundamental solution of the Laplacian on* $\mathbb{R}^4$ *is* $-1/(4\pi^2|x|^2)$.]

20. (a) State the Fourier inversion theorem for Schwartz functions $\mathcal{S}(\mathbb{R})$ on the real line. Define the Fourier transform of a tempered distribution and compute the Fourier transform of the distribution defined by the function $F(x) = \frac{1}{2}$ for $-t \le x \le +t$ and $F(x) = 0$ otherwise. (Here t is any positive number.)

Use the Fourier transform in the x variable to deduce a formula for the solution to the one dimensional wave equation

$$u_{tt} - u_{xx} = 0 \ , \qquad \text{with initial data} \qquad u(0,x) = 0 \ , \quad u_t(0,x) = g(x) \ , \qquad (*)$$

for g a Schwartz function. Explain what is meant by "finite propagation speed" and briefly explain why the formula you have derived is in fact valid for arbitrary smooth $g \in C^\infty(\mathbb{R})$.

(b) State a theorem on the representation of a smooth 2π-periodic function g as a Fourier series

$$g(x) = \sum_{\alpha \in \mathbb{Z}} \hat{g}(\alpha) e^{i\alpha x}$$

and derive a representation for solutions to $(*)$ as Fourier series in x.

(c) Verify that the formulae obtained in (a) and (b) agree for the case of smooth 2π-periodic g.

21. (i) Consider the problem of solving the equation

$$\sum_{j=1}^{n} a_j(\mathbf{x}) \frac{\partial u}{\partial x_j} = b(\mathbf{x}, u)$$

for a C^1 function $u = u(\mathbf{x}) = u(x_1, \dots, x_n)$, with data specified on a C^1 hypersurface $\mathcal{S} \subset \mathbb{R}^n$

$$u(\mathbf{x}) = \phi(\mathbf{x}), \qquad \forall \mathbf{x} \in \mathcal{S}.$$

Assume that $a_1, \dots, a_n, \phi, b$ are C^1 functions. Define the characteristic curves and explain what it means for the non-characteristic condition to hold at a point on $\mathcal{S}$. State a local existence and uniqueness theorem for the problem.

(ii) Consider the case $n = 2$ and the equation

$$\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2} = x_2 u$$

with data $u(x_1, 0) = \phi(x_1, 0) = f(x_1)$ specified on the axis $\{\mathbf{x} \in \mathbb{R}^2 : x_2 = 0\}$. Obtain a formula for the solution.

(iii) Consider next the case $n = 2$ and the equation

$$\frac{\partial u}{\partial x_1} - \frac{\partial u}{\partial x_2} = 0$$

with data $u(\mathbf{g}(s)) = \phi(\mathbf{g}(s)) = f(s)$ specified on the hypersurface $\mathcal{S}$, which is given parametrically as $\mathcal{S} \equiv \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} = \mathbf{g}(s)\}$ where $\mathbf{g} : \mathbb{R} \to \mathbb{R}^2$ is defined by

$$\mathbf{g}(s) = (s, 0), \qquad s < 0,$$

$$\mathbf{g}(s) = (s, s^2), \qquad s \ge 0.$$

Find the solution u and show that it is a global solution. (Here "global" means u is C^1 on all of $\mathbb{R}^2$.)

(iv) Consider next the equation

$$\frac{\partial u}{\partial x_1} + \frac{\partial u}{\partial x_2} = 0$$

to be solved with the same data given on the same hypersurface as in (iii). Explain, with reference to the characteristic curves, why there is generally no global C^1 solution. Discuss the existence of local solutions defined in some neighbourhood of a given point $\mathbf{y} \in \mathcal{S}$ for various $\mathbf{y}$. [*You need not give formulae for the solutions.*]

22. Define (i) the Fourier transform of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^3)$, and (ii) the convolution $T * g$ of a tempered distribution $T \in \mathcal{S}'(\mathbb{R}^3)$ and a Schwartz function $g \in \mathcal{S}(\mathbb{R}^3)$. Give a formula for the Fourier transform of $T * g$ ("convolution theorem").

    Let $t > 0$. Compute the Fourier transform of the tempered distribution $A_t \in \mathcal{S}'(\mathbb{R}^3)$ defined by

$$\langle A_t, \phi \rangle = \int_{\|y\|=t} \phi(y) d\Sigma(y), \qquad \forall\, \phi \in \mathcal{S}(\mathbb{R}^3),$$

    and deduce the Kirchhoff formula for the solution $u(t,x)$ of

$$\frac{\partial^2 u}{\partial t^2} - \Delta u = 0,$$

$$u(0,x) = 0, \qquad \frac{\partial u}{\partial t}(0,x) = g(x), \quad g \in \mathcal{S}(\mathbb{R}^3)\ .$$

    Prove, by consideration of the quantities $e = \frac{1}{2}(u_t^2 + |\nabla u|^2)$ and $p = -u_t \nabla u$, that any C^2 solution is also given by the Kirchhoff formula (uniqueness).

    Prove a corresponding uniqueness statement for the initial value problem

$$\frac{\partial^2 w}{\partial t^2} - \Delta w + V(x) w = 0,$$

$$w(0,x) = 0, \qquad \frac{\partial w}{\partial t}(0,x) = g(x), \quad g \in \mathcal{S}(\mathbb{R}^3)$$

    where V is a smooth positive real-valued function of $x \in \mathbb{R}^3$ only.

23. Write down the formula for the solution $u = u(t,x)$ for $t > 0$ of the initial value problem for the heat equation in one space dimension

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = 0\,,$$

$$u(0,x) = g(x)\,,$$

    for $g : \mathbb{R} \to \mathbb{C}$ a given smooth bounded function.

    Define the distributional derivative of a tempered distribution $T \in \mathcal{S}'(\mathbb{R})$. Define a fundamental solution of a constant-coefficient linear differential operator P, and show that the distribution defined by the function $\frac{1}{2}e^{-|x|}$ is a fundamental solution for the operator $P = -\dfrac{d^2}{dx^2} + 1$.

For the equation

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = e^t \phi(x), \qquad (*)$$

where $\phi \in \mathcal{S}(\mathbb{R})$, prove that there is a unique solution of the form $e^t v(x)$ with $v \in \mathcal{S}(\mathbb{R})$. Hence write down the solution of $(*)$ with general initial data $u(0, x) = f(x)$ and describe the large time behaviour.

24. State and prove the mean value property for harmonic functions on $\mathbb{R}^3$.

Obtain a generalization of the mean value property for sub-harmonic functions on $\mathbb{R}^3$, i.e. C^2 functions for which

$$-\Delta u(x) \leq 0$$

for all $x \in \mathbb{R}^3$.

Let $\phi \in C^2(\mathbb{R}^3; \mathbb{C})$ solve the equation

$$-\Delta\phi + iV(x)\phi = 0\,,$$

where V is a real-valued continuous function. By considering the function $w(x) = |\phi(x)|^2$ show that, on any ball $B(y, R) = \{x : \|x - y\| < R\} \subset \mathbb{R}^3$,

$$\sup_{x \in B(y,R)} |\phi(x)| \leqslant \sup_{\|x-y\|=R} |\phi(x)|.$$

25. (i) State the local existence theorem for the first order quasi-linear partial differential equation

$$\sum_{j=1}^{n} a_j(x, u)\, \frac{\partial u}{\partial x_j} \;=\; b(x, u),$$

which is to be solved for a real-valued function with data specified on a hypersurface S. Include a definition of "non-characteristic" in your answer.

(ii) Consider the linear constant-coefficient case (that is, when all the functions $a_1, \ldots, a_n$ are real constants and $b(x, u) = cx + d$ for some $c = (c_1, \ldots, c_n)$ with $c_1, \ldots, c_n$ real and d real) and with the hypersurface S taken to be the hyperplane $\mathbf{x} \cdot \mathbf{n} = 0$. Explain carefully the relevance of the non-characteristic condition in obtaining a solution via the method of characteristics.

(iii) Solve the equation

$$\frac{\partial u}{\partial y} + u\,\frac{\partial u}{\partial x} \;=\; 0,$$

with initial data $u(0, y) = -y$ prescribed on $x = 0$, for a real-valued function $u(x, y)$. Describe the domain on which your solution is C^1 and comment on this in relation to the theorem stated in (i).

26. Consider the initial value problem for the so-called Liouville equation

$$f_t + v \cdot \nabla_x f - \nabla V(x) \cdot \nabla_v f = 0,\ (x, v) \in \mathbb{R}^{2d},\ t \in \mathbb{R},$$

$$f(x, v, t = 0) = f_I(x, v),$$

for the function $f = f(x, v, t)$ on $\mathbb{R}^{2d} \times \mathbb{R}$. Assume that $V = V(x)$ is a given function with V, $\nabla_x V$ Lipschitz continuous on $\mathbb{R}^d$.

[(i)] Let $f_I(x,v) = \delta(x - x_0, v - v_0)$, for $x_0, v_0 \in \mathbb{R}^d$ given. Show that a solution f is given by

$$f(x,v,t) = \delta(x - \hat{x}(t,x_0,v_0), v - \hat{v}(t,x_0,v_0)),$$

where $(\hat{x},\hat{v})$ solve the Newtonian system

$$\dot{\hat{x}} = \hat{v}, \qquad \hat{x}(t=0) = x_0,$$
$$\dot{\hat{v}} = -\nabla V(\hat{x}), \qquad \hat{v}(t=0) = v_0.$$

[(ii)] Let $f_I \in L^1_{loc}(\mathbb{R}^{2d})$, $f_I \geq 0$. Prove (by using characteristics) that f remains non-negative (as long as it exists).

[(iii)] Let $f_I \in L^p(\mathbb{R}^{2d})$, $f_I \geq 0$ on $\mathbb{R}^{2d}$. Show (by a formal argument) that

$$\|f(\cdot,\cdot,t)\|_{L^p(\mathbb{R}^{2d})} = \|f_I\|_{L^p(\mathbb{R}^{2d})}$$

for all $t \in \mathbb{R}$, $1 \leq p < \infty$.

[(iv)] Let $V(x) = \frac{|x|^2}{2}$. Use the method of characteristics to solve the initial value problem for general initial data.

27. [(a)] Solve the initial value problem for the Burgers equation

$$u_t + \frac{1}{2}(u^2)_x = 0, \quad x \in \mathbb{R},\ t > 0,$$

$$u(x, t=0) = u_I(x),$$

where

$$u_I(x) = \begin{cases} 1, & x < 0, \\ 1 - x, & 0 < x < 1, \\ 0, & x > 1. \end{cases}$$

Use the method of characteristics. What is the maximal time interval in which this (weak) solution is well defined? What is the regularity of this solution?

[(b)] Apply the method of characteristics to the Burgers equation subject to the initial condition

$$u_I(x) = \begin{cases} 1, & x > 0, \\ 0, & x < 0. \end{cases}$$

In $\{(x,t) \,|\, 0 < x < t\}$ use the ansatz $u(x,t) = f(\frac{x}{t})$ and determine f.

[(c)] Using the method of characteristics show that the initial value problem for the Burgers equation has a classical solution defined for all $t > 0$ if u_I is continuously differentiable and

$$\frac{du_I}{dx}(x) > 0$$

for all $x \in \mathbb{R}$.

28. [(a)] Consider the nonlinear elliptic problem

$$\begin{cases} \Delta u = f(u,x), & x \in \Omega \subseteq \mathbb{R}^d, \\ u = u_D, & x \in \partial\Omega. \end{cases}$$

Let $\frac{\partial f}{\partial u}(y,x) \geq 0$ for all $y \in \mathbb{R}$, $x \in \Omega$. Prove that there exists at most one classical solution.

[*Hint: use the weak maximum principle.*]

[(b)] Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a radial function. Prove that the Fourier transform of φ is radial too.

[(c)] Let $\varphi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a radial function. Solve

$$-\Delta u + u = \varphi(x), \quad x \in \mathbb{R}^n$$

by Fourier transformation and prove that u is a radial function.

[(d)] State the Lax–Milgram lemma and explain its use in proving the existence and uniqueness of a weak solution of

$$-\Delta u + a(x)u = f(x), \ x \in \Omega,$$

$$u = 0 \ \text{ on } \partial\Omega,$$

where $\Omega \subseteq \mathbb{R}^d$ bounded, $0 \leq \underline{a} \leq a(x) \leq \overline{a} < \infty$ for all $x \in \Omega$ and $f \in L^2(\Omega)$.

29. (a) Solve by using the method of characteristics

$$x_1 \frac{\partial}{\partial x_1} u + 2\, x_2 \frac{\partial}{\partial x_2} u \;=\; 5\, u\,, \qquad u(x_1, 1) \;=\; g(x_1)\,,$$

where $g : \mathbb{R} \to \mathbb{R}$ is continuous. What is the maximal domain in $\mathbb{R}^2$ in which u is a solution of the Cauchy problem?

(b) Prove that the function

$$u(x,t) \;=\; \begin{cases} 0\,, & x < 0\,,\, t > 0\,, \\ x/t\,, & 0 < x < t\,,\, t > 0\,, \\ 1\,, & x > t > 0\,, \end{cases}$$

is a weak solution of the Burgers equation

$$\frac{\partial}{\partial t} u + \frac{1}{2}\,\frac{\partial}{\partial x} u^2 \;=\; 0\,, \qquad x \in \mathbb{R},\, t > 0\,, \tag{$*$}$$

with initial data

$$u(x,0) = \begin{cases} 0\,, & x < 0\,, \\ 1\,, & x > 0\,. \end{cases}$$

(c) Let $u = u(x,t)$, $x \in \mathbb{R}$, $t > 0$ be a piecewise C^1-function with a jump discontinuity along the curve

$$\Gamma : x \;=\; s(t)$$

and let u solve the Burgers equation $(*)$ on both sides of Γ. Prove that u is a weak solution of (1) if and only if

$$\dot{s}(t) \;=\; \frac{1}{2}\,(u_l(t) + u_r(t))$$

holds, where $u_l(t)$, $u_r(t)$ are the one-sided limits

$$u_l(t) = \lim_{x \nearrow s(t)^-} u(x,t)\,, \qquad u_r(t) = \lim_{x \searrow s(t)^+} u(x,t)\,.$$

[*Hint: Multiply the equation by a test function* $\phi \in C_0^\infty(\mathbb{R} \times [0,\infty))$, *split the integral appropriately and integrate by parts. Consider how the unit normal vector along* Γ *can be expressed in terms of* $\dot{s}$.]

30. Consider the Schrödinger equation

$$i\partial_t \Psi = -\frac{1}{2}\Delta\Psi\,, \qquad x \in \mathbb{R}^n\,,\ t > 0\,,$$

for complex-valued solutions $\Psi(x,t)$ and where Δ is the Laplacian.

(a) Derive, by using a Fourier transform and its inversion, the fundamental solution of the Schrödinger equation. Obtain the solution of the initial value problem

$$\begin{aligned} i\partial_t \Psi &= -\frac{1}{2}\Delta\Psi\,, \qquad x \in \mathbb{R}^n, \quad t > 0\,, \\ \Psi(x,0) &= f(x)\,, \qquad x \in \mathbb{R}^n\,, \end{aligned}$$

as a convolution.

(b) Consider the Wigner-transform of the solution of the Schrödinger equation

$$w(x,\xi,t) = \frac{1}{(2\pi)^n}\int_{\mathbb{R}^n} \Psi(x + \tfrac{1}{2}\,y,\, t)\, \bar{\Psi}(x - \tfrac{1}{2}\,y,\, t)\, e^{-iy\cdot\xi}\, \mathrm{d}^n y\,,$$

defined for $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$, $t > 0$. Derive an evolution equation for w by using the Schrödinger equation. Write down the solution of this evolution equation for given initial data $w(x,\xi,0) = g(x,\xi)$.

31. Solve the Dirichlet problem for the Laplace equation in a disc in $\mathbb{R}^2$

$$\begin{aligned} &\Delta u = 0 \quad \text{in} \quad G = \{x^2 + y^2 < R^2\} \subseteq \mathbb{R}^2\,,\ R > 0\,, \\ &u = u_D \quad \text{on} \quad \partial G\,, \end{aligned}$$

using polar coordinates (r,φ) and separation of variables, $u(x,y) = R(r)\Theta(\varphi)$. Then use the ansatz $R(r) = r^\alpha$ for the radial function.

32. Let $H = H(x,v)$, $x, v \in \mathbb{R}^n$, be a smooth real-valued function which maps $\mathbb{R}^{2n}$ into $\mathbb{R}$. Consider the initial value problem for the equation

$$\begin{aligned} &f_t + \nabla_v H \cdot \nabla_x f - \nabla_x H \cdot \nabla_v f = 0, \quad x, v \in \mathbb{R}^n,\ t > 0\,, \\ &f(x,v,t=0) = f_I(x,v), \quad x, v \in \mathbb{R}^n\,, \end{aligned}$$

for the unknown function $f = f(x,v,t)$.

(i) Use the method of characteristics to solve the initial value problem, locally in time.

[(ii)] Let $f_I \geq 0$ on $\mathbb{R}^{2n}$. Use the method of characteristics to prove that f remains non-negative (as long as it exists).

[(iii)] Let $F : \mathbb{R} \to \mathbb{R}$ be smooth. Prove that

$$\int_{\mathbb{R}^{2n}} F(f(x,v,t))\, dx\, dv = \int_{\mathbb{R}^{2n}} F(f_I(x,v))\, dx\, dv\,,$$

as long as the solution exists.

[(iv)] Let H be independent of x, namely $H(x,v) = a(v)$, where a is smooth and real-valued. Give the explicit solution of the initial value problem.

33. Consider the Schrödinger equation

$$\begin{aligned} i\partial_t \psi(t,x) &= -\frac{1}{2}\Delta\psi(t,x) + V(x)\psi(t,x)\;, \qquad x \in \mathbb{R}^n,\ t > 0\;, \\ \psi(t=0,x) &= \psi_I(x)\;, \qquad x \in \mathbb{R}^n\;, \end{aligned}$$

where V is a smooth real-valued function.

Prove that, for smooth solutions, the following equations are valid for all $t > 0$:

[(i)]

$$\int_{\mathbb{R}^n} |\psi(t,x)|^2\; dx = \int_{\mathbb{R}^n} |\psi_I(x)|^2\; dx\;.$$

[(ii)]

$$\begin{aligned} &\int_{\mathbb{R}^n} \frac{1}{2}|\nabla\psi(t,x)|^2\; dx + \int_{\mathbb{R}^n} V(x)|\psi(t,x)|^2\; dx \\ &= \int_{\mathbb{R}^n} \frac{1}{2}|\nabla\psi_I(x)|^2\; dx + \int_{\mathbb{R}^n} V(x)|\psi_I(x)|^2\; dx\;. \end{aligned}$$

34. [(a)] State the local existence theorem of a classical solution of the Cauchy problem

$$\begin{aligned} &a(x_1,x_2,u)\frac{\partial u}{\partial x_1} + b(x_1,x_2,u)\frac{\partial u}{\partial x_2} = c(x_1,x_2,u)\,, \\ &u|_\Gamma = u_0\,, \end{aligned}$$

where Γ is a smooth curve in $\mathbb{R}^2$.

[(b)] Solve, by using the method of characteristics,

$$\begin{aligned} &2x_1\frac{\partial u}{\partial x_1} + 4x_2\frac{\partial u}{\partial x_2} = u^2\,, \\ &u(x_1, 2) = h\,, \end{aligned}$$

where $h > 0$ is a constant. What is the maximal domain of existence in which u is a solution of the Cauchy problem?

35. Consider the functional

$$E(u) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx + \int_{\Omega} F(u, x) \, dx \,,$$

where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary and $F : \mathbb{R} \times \Omega \to \mathbb{R}$ is smooth. Assume that $F(u, x)$ is convex in u for all $x \in \Omega$ and that there is a $K > 0$ such that

$$-K \leq F(v, x) \leq K \left(|v|^2 + 1 \right) \qquad \forall v \in \mathbb{R}, \, x \in \Omega \,.$$

[(i)] Prove that E is well-defined on $H_0^1(\Omega)$, bounded from below and strictly convex. Assume without proof that E is weakly lower-semicontinuous. State this property. Conclude the existence of a unique minimizer of E.

[(ii)] Which elliptic boundary value problem does the minimizer solve?

36. Let $u_0 : \mathbb{R} \to \mathbb{R}$, $u_0 \in C^1(\mathbb{R})$, $u_0(x) \geq 0$ for all $x \in \mathbb{R}$. Consider the partial differential equation for $u = u(x, y)$,

$$4y u_x + 3u_y = u^2, \qquad (x, y) \in \mathbb{R}^2$$

subject to the Cauchy condition $u(x, 0) = u_0(x)$.

i) Compute the solution of the Cauchy problem by the method of characteristics.

ii) Prove that the domain of definition of the solution contains

$$(x, y) \in \mathbb{R} \times \left(-\infty, \frac{3}{\sup_{x \in \mathbb{R}} (u_0(x))} \right).$$

37. Consider the elliptic Dirichlet problem on $\Omega \subset \mathbb{R}^n$, Ω bounded with a smooth boundary:

$$\Delta u - e^u = f \text{ in } \Omega, \qquad u = u_D \text{ on } \partial\Omega.$$

Assume that $u_D \in L^\infty(\partial\Omega)$ and $f \in L^\infty(\Omega)$.

(i) State the strong Minimum-Maximum Principle for uniformly elliptic operators.

(ii) Prove that there exists at most one classical solution of the boundary value problem.

(iii) Assuming further that $f \geq 0$ in Ω, use the maximum principle to obtain an upper bound on the solution (assuming that it exists).

38. Consider the nonlinear partial differential equation for a function $u(x, t)$, $x \in \mathbb{R}^n$, $t > 0$,

$$u_t = \Delta u - \alpha |\nabla u|^2, \tag{6.5.9}$$
$$\text{subject to} \quad u(x, 0) = u_0(x), \tag{6.5.10}$$

where $u_0 \in L^\infty(\mathbb{R}^n)$.

(i) Find a transformation $w := F(u)$ such that w satisfies the heat equation

$$w_t = \Delta w, \quad x \in \mathbb{R}^n,$$

if (6.5.9) holds for u.

(ii) Use the transformation obtained in (i) (and its inverse) to find a solution to the initial value problem (6.5.9), (6.5.10).
[*Hint. Use the fundamental solution of the heat equation.*]

(iii) The equation (6.5.9) is posed on a bounded domain $\Omega \subseteq \mathbb{R}^n$ with smooth boundary, subject to the initial condition (6.5.10) on Ω and inhomogeneous Dirichlet boundary conditions

$$u = u_D \quad \text{on } \partial\Omega,$$

where u_D is a bounded function. Use the maximum-minimum principle to prove that there exists at most one classical solution of this boundary value problem.

39. i) State the Lax–Milgram lemma.

ii) Consider the boundary value problem

$$\begin{aligned} \Delta^2 u - \Delta u + u &= f \quad \text{in } \Omega, \\ u = \nabla u \cdot \gamma &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where Ω is a bounded domain in $\mathbb{R}^n$ with a smooth boundary, γ is the exterior unit normal vector to $\partial\Omega$, and $f \in L^2(\Omega)$. Show (using the Lax–Milgram lemma) that the boundary value problem has a unique weak solution in the space

$$H_0^2(\Omega) := \{u : \Omega \to \mathbb{R}; u = \nabla u \cdot \gamma = 0 \text{ on } \partial\Omega\}.$$

[*Hint. Show that*

$$\|\Delta u\|^2_{L^2(\Omega)} = \sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|^2_{L^2(\Omega)} \quad \textit{for all } u \in C_0^\infty(\Omega),$$

and then use the fact that $C_0^\infty(\Omega)$ *is dense in* $H_0^2(\Omega)$*. In addition, you may assume that* $H_0^2(\Omega))$ *is a Hilbert space (and so complete) with the norm*

$$\left\|u\right\|^2_{L^2(\Omega)} + \sum_{j=1}^{n} \left\| \frac{\partial u}{\partial x_j} \right\|^2_{L^2(\Omega)} + \sum_{i,j=1}^{n} \left\| \frac{\partial^2 u}{\partial x_i \partial x_j} \right\|^2_{L^2(\Omega)}$$

and that the functions in it satisfy the boundary conditions given at the beginning of the question, in an appropriate sense.]

40. (a) Discuss briefly the concept of *well-posedness* for a Cauchy problem for a partial differential equation.

Solve the Cauchy problem

$$\partial_2 u + x_1 \partial_1 u = a u^2 \,, \qquad u(x_1, 0) = \phi(x_1)$$

where $a \in \mathbb{R}$ and $\phi \in C^1(\mathbb{R})$.

For the case $a = 0$ show that the solution satisfies $\max_{x_1 \in \mathbb{R}} |u(x_1, x_2)| = \|\phi\|_{C^0}$ and deduce that the Cauchy problem is then well-posed in the uniform metric (i.e. the metric determined by the C^0 norm below).

(b) State the Cauchy-Kovalevskaya theorem and deduce that the following Cauchy problem for the Laplace equation:

$$\partial_1^2 u + \partial_2^2 u = 0\,, \qquad u(x_1, 0) = 0\,,\ \partial_2 u(x_1, 0) = \phi(x_1) \tag{6.5.11}$$

has a unique analytic solution in some neighbourhood of $x_2 = 0$ for any analytic $\phi = \phi(x_1)$. Write down the solution for the case $\phi(x_1) = \sin(nx_1)$ and hence give a sequence of initial data $\{\phi_n(x_1)\}_{n=1}^{\infty}$ with the property that

$$\|\phi_n\|_{C^r} \ \to\ 0\,, \quad \text{as } n \to \infty\text{, for each } \ r \in \mathbb{N}\,,$$

whereas u_n, the corresponding solution of (6.5.11), satisfies

$$\max_{x_1 \in \mathbb{R}} |u_n(x_1, x_2)| \ \to\ +\infty\,, \quad \text{as } n \to \infty\text{, for any } \ x_2 \neq 0\,.$$

In this question use the following definition

$$\|u\|_{C^r} = \sum_{i=0}^{r} \max_{x \in \mathbb{R}} |\partial_1^i \phi(x_1)|$$

for the C^r norm on functions $\phi = \phi(x_1)$ of one variable, with $r = 0, 1, 2\ldots$ and write $\partial_1 \phi = \partial_{x_1} \phi = \frac{\partial \phi}{\partial x_1}$ etc.

41. State the Lax-Milgram lemma.

Let $\mathbf{V} = \mathbf{V}(x_1, x_2, x_3)$ be a smooth vector field which is 2π-periodic in each co-ordinate x_j for $j = 1, 2, 3$. Formulate the definition of weak H^1_{per} solution for the equation

$$-\Delta u + \sum V_j \partial_j u + u = f \tag{6.5.12}$$

to be solved for $u = u(x_1, x_2, x_3)$ given $f = f(x_1, x_2, x_3)$ in H^0, both also 2π-periodic in each co-ordinate.

For the case that the vector field is divergence free

$$\text{div } \mathbf{V} = \sum_{j=1}^{3} \partial_j V_j = 0$$

prove that there exists a unique H^1_{per} weak solution for all such f.

For the case that $\mathbf{V}$ is the constant vector field with components $(1, 0, 0)$ write down the solution in terms of Fourier series, and show that there exists a number $C > 0$ such that

$$\|u\|_{H^2} \leq C \|f\|_{H^0}\,. \tag{6.5.13}$$

In this question use the definition

$$H^s_{per} = \{u = \sum_{m \in \mathbb{Z}^3} \hat{u}(m) e^{im\cdot x} \in L^2 : \|u\|^2_{H^s} = \sum_{m \in \mathbb{Z}^3} (1 + \|m\|^2)^s |\hat{u}(m)|^2 < \infty\}\,,$$

for the Sobolev spaces of functions 2π-periodic in each co-ordinate x_j and for $s = 0, 1, 2, \ldots$.

42. Define the *parabolic boundary* $\partial_{par}\Omega_T$ of the domain $\Omega_T = [0,1] \times (0,T]$ for $T > 0$.

Let $u = u(x,t)$ be a smooth real valued function on Ω_T which satisfies the *inequality*:

$$u_t \,-\, a u_{xx} + b u_x + c u \leq 0\,.$$

Assume the coefficients a, b and c are smooth functions and that there exist positive constants m, M such that $m \leq a \leq M$ everywhere, and $c \geq 0$. Prove that

$$\max_{(x,t)\in\overline{\Omega_T}} u(x,t) \leq \max_{(x,t)\in\partial_{par}\Omega_T} u^+(x,t)\,. \tag{6.5.14}$$

(Here $u^+ = \max\{u, 0\}$ is the positive part of the function u.)

Consider a smooth real valued function ϕ on Ω_T such that

$$\phi_t - \phi_{xx} - (1-\phi^2)\phi = 0\,, \qquad \phi(x,0) = f(x)$$

everywhere, and $\phi(0,t) = 1 = \phi(1,t)$ for all $t \geq 0$. Deduce from (6.5.14) that if $f(x) \leq 1$ for all $x \in [0,1]$ then $\phi(x,t) \leq 1$ for all $(x,t) \in \Omega_T$. *Hint: consider* $u = \phi^2 - 1$ *and compute* $u_t - u_{xx}$.

43. (i) Show that an arbitrary C^2 solution of the one dimensional wave equation $u_{tt} - u_{xx} = 0$ can be written in the form $u = F(x-t) + G(x+t)$.

Using this deduce the formula for the solution at arbitrary $t > 0$ of the Cauchy problem:

$$u_{tt} - u_{xx} = 0\,, \qquad u(0,x) = u_0(x)\,, \qquad u_t(0,x) = u_1(x)\,, \tag{6.5.15}$$

with initial data given by arbitrary Schwartz functions u_0, u_1.

Deduce from this formula a theorem on finite propagation speed for the one dimensional wave equation.

(ii) Define the Fourier transform of a tempered distribution. Compute the Fourier transform of the tempered distribution $T_t \in \mathcal{S}'(\mathbb{R})$ defined for all $t > 0$ by the function

$$T_t(y) = \begin{cases} \frac{1}{2} & \text{if } |y| \leq t, \\ 0 & \text{if } |y| > t. \end{cases}$$

(i.e. $\langle T_t\,,\, f\,\rangle = \frac{1}{2}\int_{-t}^{+t} f(y)\,dy$ for all $f \in \mathcal{S}(\mathbb{R})$). By consideration of the Fourier transform in x deduce from this the formula for the solution of (6.5.15) that you obtained in part (i) in the case $u_0 = 0$.