Asymptotic Methods: Example Sheet 3

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One more Steepest-Descent Example.

1. The Hankel function $H^{(1)}{}_{\nu}(x)$ is defined by

$$H^{(1)}{}_{\nu}(x) \quad = \quad -\frac{1}{\pi}\int_{-i\infty}^{i\infty-\pi} e^{i(\nu t - x\sin t)}\, dt \quad ,$$

where the path of integration, in the complex plane, satisfies $-\pi \leq \mathrm{Re}(t) < 0$. Use the method of steepest descent to show that, as $\nu \longrightarrow +\infty$, one has

$$H^{(1)}{}_{\nu}\left(\frac{\nu}{\cos\alpha}\right) \quad \sim \quad 2^{\frac{1}{2}}\left(\pi\nu\tan\alpha\right)^{-\frac{1}{2}} e^{i\nu(\tan\alpha-\alpha)}\, e^{-\frac{i\pi}{4}} \quad ,$$

where $0 < \alpha < \pi/2$. (This is the limit in which $\nu \longrightarrow \infty$, with x/ν a positive constant greater than 1.) Find an equation for the path along which you integrate, sketch it and justify that it is indeed the path of steepest descent.

Finding asymptotic expansion of solution to algebraic and transcendental equation.

2. Consider the equation

$$x + \sqrt{x} + 2\ln x = t + \ln t$$

(a) Show that for all $t \geq 10$ the above equation has a unique solutions $x(t)$; and that $\lim_{t\to\infty} x(t) = \infty$.

(b) Find the first 4 terms in an asymptotic approximation of $x(t)$, as $t \to \infty$.

Liouville-Green and WKBJ approximation.

3. (a) For the equation $\varepsilon^2 y'' = q(x)y$ consider solutions given formally[1] as

$$y = \exp\left(\frac{1}{\varepsilon}S_0 + S_1 + \varepsilon S_2 + \varepsilon^2 S_3 + \cdots\right).$$

for small ε. Show that the functions $\{S_n\}_{n=0}^{\infty}$ are determined iteratively by

$$(S_0')^2 = q\,, \qquad 2S_0'S_1' + S_0'' = 0\,,$$

[1] This means they should satisfy the equation to each order in ε, i.e. as formal power series in ε.

and

$$2S_0'S_n' + S_{n-1}'' + \sum_{j=1}^{n-1} S_j'S_{n-j}' = 0\,.$$

Find expressions for S_0, S_1 and S_2 in terms of q (taking care over the sign choice in S_0).

(b) Now try to apply the above solution method to the Airy equation $y'' = xy$, as follows: let [2] $\varepsilon = 1, q(x) = x$ and find a relation which determines $S_n(x)$ in terms of the functions $\{S_j(x)\}_{j=1}^{n-1}$. Show inductively that this defines a sequence of functions of the form $S_j(x) = c_j x^{\frac{3}{2}(1-j)}$ for $j \geq 2$ and some constant c_j. Deduce that Ai might reasonably be expected to admit an asymptotic expansion of the form[3]

$$\mathrm{Ai}(x) \sim Ax^{-\frac{1}{4}}\exp\left(-\frac{2}{3}x^{\frac{3}{2}}\right)\left(1 - \frac{5}{48}x^{-\frac{3}{2}} + O(\frac{1}{x^3})\right)\,, \quad (x \longrightarrow \infty)\,.$$

4. Show that, for large positive x, the equation

$$w'' - x^3 w' + x^{-2} w = 0$$

has two independent solutions with asymptotic behaviour

$$w \sim 1 + O\big(x^{-4}\big) \qquad \text{and} \qquad w \sim x^{-3}\exp\left(\frac{x^4}{4}\right)\Big(1 + O\big(x^{-4}\big)\Big)\,.$$

5. (a) Consider the differential equation

$$z\,w'' - z\,w' - w = 0 \tag{1}$$

in a neighbourhood of the point at infinity, $z = \infty$. Classify this point.

(b) Show that one solution of (1) is $w(z) = ze^z$, and hence find an expression for a second independent solution as an integral.

(c) Find $\mu(z)$ and $q(z)$ such that $W(z) = \mu(z)^{-1}\,w(z)$ satisfies an equation $W'' - q(z)\,W = 0$. Write down expressions for the two independent Liouville–Green solutions of this equation in terms of $q(z)$. Calculate the first two terms of the expansion in inverse powers of z of $\sqrt{q(z)}$ about $z = \infty$, and hence determine from the Liouville–Green solutions the leading–order asymptotic form for w as $z \longrightarrow \infty$. Compare these with the two exact solutions obtained previously.

[2] There are two ways you can think of the procedure of introducing the small parameter ε and then putting it equal to one. Either ε arises as a small parameter by first rescaling $x = \mu X$ with $\mu \to +\infty$, deriving the asymptotic relations for the $\{S_j(X)\}$, and noticing that they are in fact *scale invariant* so one can rescale back. Alternatively, just think of it as a device to obtain a system for the $\{S_j\}$ that is *formally completely equivalent to the original equation* - in this case, the asymptotic content arises in checking that the solutions of the system form an asymptotic sequence, as you are asked to do here.

[3] You are not expected to prove that this procedure gives a complete asymptotic expansion. The integral formulation for the Airy functions provides an efficient basis for a rigorous approach to their asymptotics. See Section 7.6 in Hormander's "Analysis of Linear Partial Differential Operators I" or the appendix to "Complex Analysis" by Stein and Shakarchi for some proofs.

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6. The wavefunction ψ obeys the equation

$$-\frac{d^2\psi}{dx^2} - \frac{s(s+1)}{\cosh^2 x}\,\psi \quad = \quad E\,\psi\,,$$

with $s > 0$. Use the WKBJ approximation to find approximate values for the bound-state energies (eigenvalues E for which $\psi \to 0$ as $|x| \to \infty$)

$$\sqrt{(-E)} \quad = \quad \sqrt{s(s+1)} \;-\; n \;-\; \frac{1}{2} \quad .$$

What are the allowed values of n? Compare with the exact bound state energies $E = -(s-n)^2$, where $n = 0, 1, 2, \ldots$ and $n < s$.

[**Hint:** You may find the substitution $\sinh x = z$ useful, and note that if a, b are real,

$$\int_{-a}^{a} \frac{\sqrt{a^2 - z^2}}{b^2 + z^2}\,dz = \pi(\sqrt{1 + b^{-2}a^2} - 1)\,,$$

which can be derived via contour integration.]

7. Show that the large eigenvalues of the system

$$\frac{d^2w}{dx^2} \quad = \quad \left(x^4 + x^2 - \lambda^2\right) w \;, \qquad\qquad w(-\infty) = w(\infty) = 0\,,$$

are given by

$$\lambda \quad \sim \quad 2^{\frac{1}{3}}\,\pi\left(\Gamma\left(\frac{1}{4}\right)\right)^{-\frac{4}{3}}\left(3n + \frac{3}{2}\right)^{\frac{2}{3}} + O\left(n^{\frac{1}{3}}\right),$$

where n is a positive integer.

8. ($*$) Quantum tunneling through a potential barrier in the stationary approach to scattering theory is described by the time-independent Schrödinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi = E\psi$$

where $V \geq 0$ and $V \to 0$ faster than $1/|x|$ as $|x| \to \infty$. Consider a particle incident from $x = -\infty$ with an energy $E = \frac{\hbar^2 k^2}{2m}$, and assume that $E < V$ on an interval (a, b) and $E > V$ on $(-\infty, a)$ and (b, ∞). This means that for $x \to -\infty$ the wave function should approach $e^{ikx} + re^{-ikx}$ while for $x \to +\infty$ the wave function should approach te^{ikx}, where the constants r and t are known as the reflection and transmission coefficients of the potential. (The term e^{ikx} corresponds to the incident wave, travelling to the right from $x = -\infty$ in the corresponding time-dependent description.)

Write down Liouville-Green approximations to the wave functions in the three regions (a, b), $(-\infty, a)$ and (b, ∞), consistent with the set-up just described and indicating clearly which are oscillatory and which are real exponential solutions. Show using the connection formulae of the WKBJ approximation that the transmission probability $|t|^2$ is approximately

$$\exp\left(-\frac{2}{\hbar}\int_a^b \sqrt{2m(V(x) - E)}\,dx\right)\,,$$

when the integral is large compared with $\hbar$.

[**Note:** For a careful treatment see Landau and Lifshitz, Quantum Mechanics, 3rd ed., p.178, or Bender and Orszag p.524.]

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9. The analytic function $f(z)$ is defined by

$$f(z) \quad = \quad \int_{i\infty}^{z} \exp\bigl(s^2\bigr)\, ds\ .$$

Use the method of integration by parts to obtain an asymptotic expansion of the form

$$f(z) \quad \sim \quad \exp\bigl(z^2\bigr) \left\{ \frac{1}{2z} + \frac{1}{4z^3} + \frac{3}{8z^5} + \frac{15}{16z^7} + \ \dots \right\},$$

in the sector $0 \le \arg z \le \pi$, including an expression for the remainder after a finite number of terms and a verification of the asymptotic property.

To deal with the case $-\pi \le \arg z \le 0$, convert $f(z)$ to the form

$$f(z) \quad = \quad \int_{-i\infty}^{z} \exp\bigl(s^2\bigr)\, ds \ - \ i\,\pi^{\frac{1}{2}}\ ,$$

and, without making detailed calculations, obtain an asymptotic expansion valid in this domain for $\Bigl(f(z) + i\,\pi^{\frac{1}{2}}\Bigr)$. Find the Stokes lines for these asymptotic expansions of f as $|z| \to \infty$.

10. $(*)$ In discussions of the methods of stationary phase and steepest descent, we have considered the integral

$$I(z) \ = \ \int_0^1 \exp\bigl(z\,t^3\bigr)\, dt\,,$$

in the case that $z = ix$ is purely imaginary. This question is concerned with the general case $z \in \mathbb{C}$.

(a) In the case that $\Re z > 0$ show that

$$I(z) + \frac{\Gamma\bigl(\frac{4}{3}\bigr)}{z^{\frac{1}{3}}} \sim \frac{e^z}{3} \sum_{n=0}^{\infty} \frac{\Gamma\bigl(n+\frac{2}{3}\bigr)}{\Gamma\bigl(\frac{2}{3}\bigr)} \frac{1}{z^{n+1}} \qquad (\dagger)$$

as $z \longrightarrow +\infty$ in the sector $\operatorname{Arg} z \in (-\frac{\pi}{2}+\delta,\, \frac{\pi}{2}-\delta)$ for small positive δ, and obtain a similar relation for the sector $\operatorname{Arg} z \in (\frac{\pi}{2}+\delta,\, \frac{3\pi}{2}-\delta)$.

(b) Generalize the application of the method of steepest descent in the notes to develop the asymptotic expansion

$$I(z) \sim \frac{e^z}{3} \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{2}{3})}{\Gamma(\frac{2}{3})} \frac{1}{z^{n+1}} + \frac{\Gamma\bigl(\frac{4}{3}\bigr)\, e^{\frac{i\pi}{3}}}{z^{\frac{1}{3}}}\,,$$

as $|z| \to \infty$ in the sector $\operatorname{Arg} z \in (\delta,\, \pi-\delta)$. Find a similar relation for the sector $\operatorname{Arg} z \in (\pi+\delta,\, 2\pi-\delta)$. Discuss the relation between the different expansions in regions of overlap. [You may use the complex version of Watson's lemma; see Section IV.4 in Stuart's notes.]

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