MATHEMATICAL TRIPOS **H. Osborn**

PART II: Alternatives A & B ho@damtp.cam.ac.uk

Foundations of Quantum Mechanics - Supplementary past tripos problems

94116. Let $\hat{x}, \hat{p}$ and $\hat{H} = \hat{p}^2/(2m) + V(\hat{x})$ be the position operator, momentum operator and Hamiltonian of a quantum mechanical particle moving in one dimension. Let $|\Psi\rangle$ be a state vector for the particle. The position and momentum eigenstates are connected by

$$\langle x|p\rangle = \sqrt{\frac{1}{2\pi\hbar}}\, e^{ipx/\hbar}\,,$$

(where $\langle x|x'\rangle = \delta(x-x')$ and $\langle p|p'\rangle = \delta(p-p')$). Using the standard methods of the Dirac formulation of quantum mechanics, show that

$$\langle x|\hat{p}|x'\rangle = -i\hbar\,\frac{\partial}{\partial x}\,\delta(x-x') \qquad \text{and} \qquad \langle p|\hat{x}|p'\rangle = i\hbar\,\frac{\partial}{\partial p}\,\delta(p-p').$$

Find $\langle x|\hat{H}|x'\rangle$. Express $\langle x|\hat{p}|\Psi\rangle$ and $\langle x|\hat{H}|\Psi\rangle$ in terms of the position-representation wave-function for the particle.
Obtain the momentum-representation form of the Schrödinger equation of the simple harmonic oscillator. Show that, if the variable y is defined by $p = m\omega y$ (where ω is the frequency of the oscillator), then it is formally identical, as a differential equation, to the position-representation Schrödinger equation.

94215. Give a brief account of the treatment in perturbation theory of a degenerate energy level.
The unperturbed Hamiltonian H_0 of two independent one-dimensional operators is

$$H_0 = a^\dagger a + 2b^\dagger b,$$

where a and b are operators such that $[a, a^\dagger] = 1,\ [b, b^\dagger] = 1$. Find the degeneracies of the eigenvalues of H_0 with energies $E_0 = 0, 1, 2, 3, 4$.
Now consider the perturbed Hamiltonian

$$H = H_0 + fG,$$

where f is a small constant and

$$G = (a^\dagger)^2 b + a^2 b^\dagger.$$

Calculate the splitting of the level with $E_0 = 2$, to lowest order in f.
Show that $[H_0, G] = 0$. Hence, by constructing eigenstates of H as linear combinations of the eigenstates of H_0 with $E_0 = 2$, show that the result to order f is exact.

94420. Let a_1, a_2 be a pair of independent annihilation operators, obeying

$$[a_i, a_j{}^\dagger] = \delta_{ij} \quad (i, j = 1, 2).$$

Show that the operators

$$J_+ = a_1{}^\dagger a_2, \quad J_- = a_1 a_2{}^\dagger, \quad J_3 = \tfrac{1}{2}(a_1{}^\dagger a_1 - a_2{}^\dagger a_2)\,,$$

where $J_\pm = J_1 \pm iJ_2$, obey the standard angular momentum commutation relations. Define

$$|j\ m\rangle = [(j+m)!(j-m)!]^{-\frac{1}{2}}(a_1{}^\dagger)^{j+m}(a_2{}^\dagger)^{j-m}|0\rangle\,,$$

where $|0\rangle$ satisfies $a_1|0\rangle = 0,\ a_2|0\rangle = 0$. Verify that J_+, J_-, J_3 defined above have the standard actions on $|j\ m\rangle$. Write down $\mathbf{J}^2$ in terms of J_+, J_- and J_3 and show that

$$\mathbf{J}^2|j\ m\rangle = j(j+1)|j\ m\rangle.$$

Show that the operator $U(\theta) = \exp(-iJ_2\theta)$ is unitary. Let

$$f_i(\theta) = U(\theta) a_i U(\theta)^{-1} \quad (i = 1, 2).$$

Find differential equations obeyed by $f_i(\theta)$ and hence show that

$$f_1(\theta) = a_1 \cos \tfrac{1}{2}\theta + a_2 \sin \tfrac{1}{2}\theta\,, \qquad f_2(\theta) = -a_1 \sin \tfrac{1}{2}\theta + a_2 \cos \tfrac{1}{2}\theta\,.$$

Thus show that

$$U(\pi)|j\ m\rangle = (-1)^{j-m}|j\ {-m}\rangle \qquad \text{and} \qquad U(2\pi)|j\ m\rangle = (-1)^{2j}|j\ m\rangle.$$

[*Units are used in which* $\hbar = 1$. *Properties of the quantum simple harmonic oscillator may be assumed without proof.*]

91129. Two spin $\frac{1}{2}$ particles are described by Pauli matrices $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, [\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2] = 0$ and the total spin is $\mathbf{S} = \frac{1}{2}(\boldsymbol{\sigma}_1 + \boldsymbol{\sigma}_2)$. Construct the states with spin $S = 1, S_z = 1, 0, -1$ and also with $S = 0$. Assume the system has a Hamiltonian $H = \frac{1}{2}\lambda\hbar(\sigma_{1x} - \sigma_{2x})$ and that it is in the state $S = S_z = 1$ at $t = 0$. What is the probability that the system is in the state $S = 0$ at a later time t?
[*It may be convenient to use the notation that the eigenvectors of* σ_{1z} *are written as*

$$\alpha_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad \beta_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and correspondingly for σ_{2z} *are* α_2, β_2 *and to verify with this notation*

$$\alpha_1\alpha_2 = \tfrac{1}{4}\left((\alpha_1+\beta_1)(\alpha_2-\beta_2) + (\alpha_1+\beta_1)(\alpha_2+\beta_2) + (\alpha_1-\beta_1)(\alpha_2-\beta_2) + (\alpha_1-\beta_1)(\alpha_2+\beta_2)\right),$$

where each term in an eigenvector of $\sigma_{1x} - \sigma_{2x}$.]

92429. A Hamiltonian H has eigenvalues E_n and corresponding non-degenerate eigenstates $|\psi_n\rangle$. Show that under a small change of δH.

$$\delta|\psi_n\rangle = \sum_{m \neq n} \frac{\langle\psi_m|\delta H|\psi_n\rangle}{E_n - E_m}\,|\psi_m\rangle,$$

and derive the related formula for δE_n to first and second order in δH.
Assume $H = H_0 + \lambda V$, for λ a continuous parameter, has energy eigenvalues $E_n(\lambda)$ and eigenstates $|\psi_n(\lambda)\rangle$. Let $V_{nm}(\lambda) = \langle\psi_n(\lambda)|V|\psi_m(\lambda)\rangle$. Show that

$$\frac{d}{d\lambda}E_n = V_{nn}\,, \quad \frac{d^2}{d\lambda^2}E_n = -2\sum_{m\neq n}\frac{|V_{nm}|^2}{E_m - E_n}\,, \quad \frac{d}{d\lambda}V_{nm} = \sum_{r\neq m}\frac{V_{nr}V_{rm}}{E_m - E_r} + \sum_{r\neq n}\frac{V_{nr}V_{rm}}{E_n - E_r}\,.$$

Restrict these equations to two states, $n = 1, 2$ and let $x = E_1 - E_2,\ y = V_{12} = V_{21}$. Show that then

$$\frac{d^2}{d\lambda^2}x = \frac{4y^2}{x}\,, \qquad \frac{d}{d\lambda}(xy) = 0,$$

and hence show that $x = x_0(1+\lambda^2)^{\frac{1}{2}},\ y = \frac{1}{2}x_0(1+\lambda^2)^{-\frac{1}{2}}$ is a possible solution.
Describe briefly how the solution is relevant for the Hamiltonian $H = \sigma_3 + \lambda\sigma_1$, where σ_3, σ_1 are Pauli matrices.

88431. $J_\pm, J_3$ are angular momentum operators obeying

$$[J_3, J_\pm] = \pm J_\pm\,, \quad [J_+, J_-] = 2J_3\,, \quad {J_3}^\dagger = J_3\,, \quad {J_+}^\dagger = J_-\,.$$

If $C = J_3^2 + \frac{1}{2}(J_+J_- + J_-J_+)$ show that $[J_3, C] = [J_\pm, C] = 0$. Let $|j\,m\rangle$ be normalized eigenvectors of C, J_3, with eigenvalues $j(j+1), m$. Show that, with suitable conventions,

$$J_\pm|j\,m\rangle = \left\{(j+\tfrac{1}{2})^2 - (m \pm \tfrac{1}{2})^2\right\}^{\frac{1}{2}}|j\,m{\pm}1\rangle\,.$$

Hence determine the allowed values of j, m.
Consider now operators obeying

$$[\Gamma_3, \Gamma_\pm] = \pm\Gamma_\pm\,, \quad [\Gamma_+, \Gamma_-] = -2\Gamma_3\,, \quad {\Gamma_3}^\dagger = \Gamma_3\,, \quad {\Gamma_+}^\dagger = \Gamma_-\,. \qquad (*)$$

Show that $[\Gamma_3, D] = [\Gamma_\pm, D] = 0$ where $D = {\Gamma_3}^2 - \frac{1}{2}(\Gamma_+\Gamma_- + \Gamma_-\Gamma_+)$. Let $|j\,m\rangle$ now be eigenvectors of D, Γ_3, with eigenvalues $j(j+1), m$. Verify that the eigenvalues of $\Gamma_+|j\,m\rangle$ are $j(j+1), m+1$. Calculate the norm of $\Gamma_+|j\,m\rangle$ in terms of the norm of $|j\,m\rangle$ and show that for it to be positive it is sufficient that

$$-m - 1 < j < m.$$

Show also that $\Gamma_-|j\,m_0\rangle = 0$ for $m_0 = -j$ or $m_0 = j+1$. Hence demonstrate that for each $j, -\frac{1}{2} < j < 0$, there are two sets of states each linked by $\Gamma_\pm$, for each of which there is a minimum value of m but no maximum value.
Let $a, a^\dagger$ be harmonic oscillator annihilation and creation (ladder) operators. Show that

$$\Gamma_+ = \tfrac{1}{2}(a^\dagger)^2\,, \quad \Gamma_- = \tfrac{1}{2}a^2\,, \quad \Gamma_3 = \tfrac{1}{4}(a^\dagger a + a a^\dagger)$$

obey $(*)$. Find those states $|\psi\rangle$ in this case such that $\Gamma_-|\psi\rangle = 0$, and from the eigenvalues of Γ_3 determine the appropriate values for j.