Principles of Quantum Mechanics - Problems 1

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk $(*)$ *may be more difficult.*

---

1. Suppose a non–Hermitian operator A obeys $[A, A^\dagger] = 0$. What can be said about the relation between the eigenvalues of A and those of $A^\dagger$? What can be said about the inner product of two eigenstates of A with different eigenvalues?

2. If A and B are any operators which each commute with $[A, B]$, prove that $[A, B^n] = nB^{n-1}[A, B]$ and that $[A, e^B] = e^B[A, B]$. Define an operator–valued function of $\lambda \in \mathbf{C}$ by $F(\lambda) := \mathrm{e}^{\lambda A}\,\mathrm{e}^{\lambda B}\,\mathrm{e}^{-\lambda(A+B)}$. Show that $F'(\lambda) = \lambda\,[A, B]\,F(\lambda)$ and hence deduce that
$$\mathrm{e}^A\,\mathrm{e}^B = \mathrm{e}^{A+B+\frac{1}{2}[A,B]} = \mathrm{e}^B\,\mathrm{e}^A\,\mathrm{e}^{[A,B]}\,.$$

3. Prove that for any two operators A and B,
$$\frac{d}{d\lambda}\left(\mathrm{e}^{\lambda A}B\mathrm{e}^{-\lambda A}\right) = \mathrm{e}^{\lambda A}\,[A, B]\,\mathrm{e}^{-\lambda A}\,.$$
By repeated use of this relation, express $\mathrm{e}^{\lambda A}B\,\mathrm{e}^{-\lambda A}$ a series expansion in λ and deduce that
$$\mathrm{e}^A B\,\mathrm{e}^{-A} = B + [A, B] + \frac{1}{2}\,[A, [A, B]] + \cdots\,.$$

4. An operator G is defined in terms of the parity operator $\mathtt{P}$ by
$$G := \frac{1}{2}\,(1 - \mathtt{P})\,.$$
Find an expression for G^2 and hence show that $\mathrm{e}^{\mathrm{i}\pi G} = \mathtt{P}$.

5. For the isotropic quantum harmonic oscillator with Hamiltonian
$$H = \frac{\mathbf{P}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{X}^2\,,$$
show that the position and momentum operators in the Heisenberg picture are given by
$$\mathbf{X}(t) = \mathbf{X}\,\cos(\omega t) + \frac{1}{m\omega}\,\mathbf{P}\,\sin(\omega t)$$
$$\mathbf{P}(t) = \mathbf{P}\,\cos(\omega t) - m\omega\,\mathbf{X}\,\sin(\omega t)\,.$$
where $\mathbf{X}(0)$ and $\mathbf{P}(0)$ coincide with the Schrödinger picture operators. Interpret this result. Evaluate $[\mathbf{X}(t), \mathbf{P}(t)]$.

6. Show that
$$P_r := \frac{1}{2}(\hat{\mathbf{X}}\cdot\mathbf{P} + \mathbf{P}\cdot\hat{\mathbf{X}}) = \hat{\mathbf{X}}\cdot\mathbf{P} - \frac{\mathrm{i}\hbar}{|\mathbf{X}|}\,.$$
Evaluate the commutators $[|\mathbf{X}|, P_r]$ and $[\mathbf{a}\times\hat{\mathbf{X}}, P_r]$, where $\mathbf{a}$ is a constant vector. What is the interpretation of P_r?

By writing $\mathbf{L}^2 = (\mathbf{X}\times\mathbf{P})\cdot(\mathbf{X}\times\mathbf{P})$, show that
$$\mathbf{P}^2 = \frac{1}{\mathbf{X}^2}\left[\mathbf{L}^2 + (\mathbf{X}\cdot\mathbf{P})^2 - \mathrm{i}\hbar\,\mathbf{X}\cdot\mathbf{P}\right] = \frac{\mathbf{L}^2}{\mathbf{X}^2} + P_r^2\,.$$
Obtain an expression for P_r in the position representation, and thus give a relation between $\mathbf{L}^2$ in the position representation and the Laplacian ∇^2.

7. Galilean boosts with fixed velocity $\mathbf{v}$ act on Hilbert space through a unitary operator $U(\mathbf{v})$ defined by
$$U^{-1}(\mathbf{v})\,\mathbf{X}(t)\,U(\mathbf{v}) = \mathbf{X}(t) + \mathbf{v}t\,, \qquad (\star)$$
where $\mathbf{X}(t)$ is the position operator in the Heisenberg picture. Explain why the generators $\mathbf{K}$ of this boost obey $[K_i, K_j] = 0$ and write down the result of the commutator $[J_i, K_j]$ with the angular momentum operator. Show that the transformation $(\star)$ requires that the Hamiltonian obeys
$$U^{-1}(\mathbf{v})\,H\,U(\mathbf{v}) = H + \mathbf{v}\cdot\mathbf{P}\,.$$
Hence evaluate the commutators $[\mathbf{K}, H]$.

How should $U(\mathbf{v})$ act on the momentum operator $\mathbf{P}(t)$ in the Heisenberg picture? Evaluate the commutators $[K_i, P_j]$.

A set of n mutually interacting particles have Hamiltonian
$$H = \sum_{a=1}^{n} \frac{\mathbf{P}_a^2}{2m_a} + V(\mathbf{X}_a)$$
where $(\mathbf{X}_a, \mathbf{P}_a)$ are the position and momentum operators for the a^{th} particle. What condition does your result for $[\mathbf{K}, H]$ place on the potential $V(\mathbf{X}_a)$? [*Assume that the boost operators act equally on all n particles.*]

8. An electron moves along an infinite chain of potential wells. For sufficiently low energies we can assume that the set $\{|n\rangle\}$ is complete, where $|n\rangle$ is the state of definitely being in the n^{th} well. Assume that the only non-vanishing matrix elements of the Hamiltonian

are $\mathcal{E} := \langle n|H|n\rangle$ and $A := \langle n \pm 1|H|n\rangle$. Give an interpretation of $\mathcal{E}$ and A. What does the assumption that $\langle n + r|H|n\rangle$ are negligible for $|r| > 1$ mean?

Expanding an energy eigenstate $|E\rangle$ in this basis as $|E\rangle = \sum_n c_n|n\rangle$, show that

$$c_m(E - \mathcal{E}) - A(c_{m+1} - c_{m-1}) = 0\,.$$

Obtain solutions of these equations in which $c_m \propto \mathrm{e}^{\mathrm{i}km}$ and thus find the corresponding energies E_k. Why is there an upper limit of the values of k that need be considered?

Initially, the electron is in the state $|\psi\rangle = (|E_k\rangle + |E_{k+\Delta}\rangle)/\sqrt{2}$, where $0 < \Delta \ll k \ll 1$. Describe the electron's subsequent motion.