Principles of Quantum Mechanics - Problems 1

Please email me with any comments about these problems, particularly if you spot an error. Problems with an asterisk (∗) may be more difficult.

1. Suppose a non–Hermitian operator A obeys $[A, A^\dagger] = 0$. What can be said about the relation between the eigenvalues of A and those of $A^\dagger$? What can be said about the inner product of two eigenstates of A with different eigenvalues?

2. If A and B are any operators which each commute with $[A, B]$, prove that $[A, B^n] = nB^{n-1}[A, B]$ and that $[A, e^B] = e^B[A, B]$. Define an operator–valued function of $\lambda \in \mathbb{C}$ by $F(\lambda) := \mathrm{e}^{\lambda A}\,\mathrm{e}^{\lambda B}\,\mathrm{e}^{-\lambda(A+B)}$. Show that $F'(\lambda) = \lambda\,[A, B]\,F(\lambda)$ and hence deduce that
$$\mathrm{e}^{A}\,\mathrm{e}^{B} = \mathrm{e}^{A+B+\frac{1}{2}[A,B]} = \mathrm{e}^{B}\,\mathrm{e}^{A}\,\mathrm{e}^{[A,B]}\,.$$

3. Prove that for any two operators A and B,
$$\frac{d}{d\lambda}\left(\mathrm{e}^{\lambda A}B\mathrm{e}^{-\lambda A}\right) = \mathrm{e}^{\lambda A}\,[A, B]\,\mathrm{e}^{-\lambda A}\,.$$
By repeated use of this relation, express $\mathrm{e}^{\lambda A}B\,\mathrm{e}^{-\lambda A}$ a series expansion in λ and deduce that
$$\mathrm{e}^{A}B\,\mathrm{e}^{-A} = B + [A, B] + \frac{1}{2}\,[A, [A, B]] + \cdots\,.$$

4. An operator G is defined in terms of the parity operator $\mathtt{P}$ by
$$G := \frac{1}{2}\,(1 - \mathtt{P})\,.$$
Find an expression for G^2 and hence show that $\mathrm{e}^{\mathrm{i}\pi G} = \mathtt{P}$.

5. For the isotropic quantum harmonic oscillator with Hamiltonian
$$H = \frac{\mathbf{P}^2}{2m} + \frac{1}{2}m\omega^2\mathbf{X}^2\,,$$
show that the position and momentum operators in the Heisenberg picture are given by
$$\mathbf{X}(t) = \mathbf{X}\,\cos(\omega t) + \frac{1}{m\omega}\,\mathbf{P}\,\sin(\omega t)$$
$$\mathbf{P}(t) = \mathbf{P}\,\cos(\omega t) - m\omega\,\mathbf{X}\,\sin(\omega t)\,.$$
where $\mathbf{X}(0)$ and $\mathbf{P}(0)$ coincide with the Schrödinger picture operators. Interpret this result. Evaluate $[\mathbf{X}(t), \mathbf{P}(t)]$.

6. We define a function $f(Q)$ of an operator Q by $f(Q) = \int f(q)\,|q\rangle\langle q|\,dq$. Show that
$$U^{-1}(\mathbf{a})\,f(\mathbf{X})\,U(\mathbf{a}) = f(\mathbf{X}+\mathbf{a})$$
where $U(\mathbf{a})$ is the translation operator. Hence show that
$$P_r := \frac{1}{2}(\hat{\mathbf{X}}\cdot\mathbf{P} + \mathbf{P}\cdot\hat{\mathbf{X}}) = \hat{\mathbf{X}}\cdot\mathbf{P} - \frac{\mathrm{i}\hbar}{|\mathbf{X}|}$$
where $\hat{\mathbf{X}} = \mathbf{X}/|\mathbf{X}|$ and $|\mathbf{X}| = \sqrt{\mathbf{X}\cdot\mathbf{X}}$. Evaluate the commutators $[|\mathbf{X}|, P_r]$ and $[\mathbf{a}\times\hat{\mathbf{X}}, P_r]$, where $\mathbf{a}$ is a constant vector. What is the interpretation of P_r?

By writing $\mathbf{L}^2 = (\mathbf{X}\times\mathbf{P})\cdot(\mathbf{X}\times\mathbf{P})$, show that
$$\mathbf{P}^2 = \frac{1}{\mathbf{X}^2}\left[\mathbf{L}^2 + (\mathbf{X}\cdot\mathbf{P})^2 - \mathrm{i}\hbar\,\mathbf{X}\cdot\mathbf{P}\right] = \frac{\mathbf{L}^2}{\mathbf{X}^2} + P_r^2\,.$$
Obtain an expression for P_r in the position representation, and thus give a relation between $\mathbf{L}^2$ in the position representation and the Laplacian ∇^2.

7. Galilean boosts with fixed velocity $\mathbf{v}$ act on Hilbert space through a unitary operator $U(\mathbf{v})$ defined by
$$\begin{aligned} U^{-1}(\mathbf{v})\,\mathbf{X}(t)\,U(\mathbf{v}) &= \mathbf{X}(t) + \mathbf{v}t \\ U^{-1}(\mathbf{v})\,\mathbf{P}(t)\,U(\mathbf{v}) &= \mathbf{P}(t) + m\mathbf{v}\,, \end{aligned} \qquad (\star)$$
where $\mathbf{X}(t)$ and $\mathbf{P}(t)$ are the position and momentum operators in the Heisenberg picture and m is the mass of the particle. Evaluate the commutators $[K_i, P_j]$ and $[K_i, X_j]$ and show that $(\star)$ implies $[K_i, K_j] = 0$.

Show that the transformations in $(\star)$ require that the Hamiltonian obeys
$$U^{-1}(\mathbf{v})\,H\,U(\mathbf{v}) = H + \mathbf{v}\cdot\mathbf{P} + f(\mathbf{v})$$
for some function $f(\mathbf{v})$ that is independent of the position and momentum operators, and fix this function using your results for successive boosts. Evaluate the commutators $[K_i, H]$.

A set of n mutually interacting particles have Hamiltonian
$$H = \sum_{a=1}^{n} \frac{\mathbf{P}_a^2}{2m_a} + V(\mathbf{X}_a)$$
where $(\mathbf{X}_a, \mathbf{P}_a)$ are the position and momentum operators for the a^{th} particle. What condition does your result for $[\mathbf{K}, H]$ place on the potential $V(\mathbf{X}_a)$? [*Assume that the boost operators act equally on all n particles.*]

8. An electron moves along an infinite chain of potential wells. For sufficiently low energies we can assume that the set $\{|n\rangle\}$ is complete, where $|n\rangle$ is the state of definitely being in the n^{th} well. Assume that the only non-vanishing matrix elements of the Hamiltonian are $\mathcal{E} := \langle n|H|n\rangle$ and $A := \langle n \pm 1|H|n\rangle$. Give an interpretation of $\mathcal{E}$ and A. What does the assumption that $\langle n + r|H|n\rangle$ are negligible for $|r| > 1$ mean?

   Expanding an energy eigenstate $|E\rangle$ in this basis as $|E\rangle = \sum_n c_n|n\rangle$, show that

   $$c_m(E - \mathcal{E}) - A(c_{m+1} + c_{m-1}) = 0\,.$$

   Obtain solutions of these equations in which $c_m \propto \mathrm{e}^{ikm}$ and thus find the corresponding energies E_k. Why is there an upper limit of the values of k that need be considered?

   Initially, the electron is in the state $|\psi\rangle = (|E_k\rangle + |E_{k+\Delta}\rangle)/\sqrt{2}$, where $0 < \Delta \ll k \ll 1$. Describe the electron's subsequent motion.