

Principles of Quantum Mechanics - Problems 2

Please *email me* with any comments about these problems, particularly if you spot an error.

1. Let \mathbf{n} be the unit vector in the direction with polar coordinates (θ, ϕ) . Write down the matrix $\mathbf{n} \cdot \boldsymbol{\sigma}$ and find its eigenvectors. Hence show that the state of a spin- $\frac{1}{2}$ particle in which a measurement of the component of spin along \mathbf{n} is certain to yield $\hbar/2$ is

$$|\uparrow_{\mathbf{n}}\rangle = \sin \frac{\theta}{2} e^{i\phi/2} |\downarrow\rangle + \cos \frac{\theta}{2} e^{-i\phi/2} |\uparrow\rangle. \quad (\star)$$

where $|\uparrow\rangle, |\downarrow\rangle$ are the usual eigenstates of S_z . Obtain the corresponding expression for $|\downarrow_{\mathbf{n}}\rangle$. Explain why each of the coefficients in (\star) has modulus $1/\sqrt{2}$ when $\theta = \pi/2$, and why $\langle \uparrow | \uparrow_{\mathbf{n}} \rangle = 0$ at $\theta = \pi$.

2. Write down the 3×3 matrix that represents S_x for a spin-1 system in the basis in which $S_z = \text{diag}(\hbar, 0, -\hbar)$.

An unpolarized beam of spin-1 particles enters a Stern–Gerlach filter that passes only particles with $S_z = \hbar$. On exiting this filter, the beam enters a second filter that passes only particles with $S_x = \hbar$ and then finally it encounters a filter that passes only particles with $S_z = -\hbar$. What fraction of the initial particles make it right through?

3. Consider the addition of angular momentum 3 to angular momentum 1. Express the states $|j, m\rangle \in \{|4, 4\rangle, |4, 3\rangle, |3, 3\rangle, |3, 2\rangle, |2, 2\rangle, |2, 1\rangle\}$ in terms of the states of the subsystems.
4. The interaction between neighbouring spin- $\frac{1}{2}$ atoms in a certain crystal is described by the Hamiltonian

$$H = K \left(\frac{\mathbf{S}^{(1)} \cdot \mathbf{S}^{(2)}}{|\mathbf{a}|} - 3 \frac{(\mathbf{a} \cdot \mathbf{S}^{(1)})(\mathbf{a} \cdot \mathbf{S}^{(2)})}{|\mathbf{a}|^3} \right),$$

where \mathbf{a} is the separation between the atoms, K is a constant and $\mathbf{S}^{(1)}$ is the first atom's spin operator. Explain the physical idea underlying this form of H .

Let $\mathbf{S} = \mathbf{S}^{(1)} + \mathbf{S}^{(2)}$ be the total spin operator. Show that we can find a complete set of mutual eigenstates of $\mathbf{S} \cdot \mathbf{S}$, $\mathbf{a} \cdot \mathbf{S}$ and H . By showing that

$$S_x^{(1)} S_x^{(2)} + S_y^{(1)} S_y^{(2)} = \frac{1}{2} \left(S_+^{(1)} S_-^{(2)} + S_-^{(1)} S_+^{(2)} \right),$$

find the energy eigenvalues. [*Make an appropriate choice for the direction of $\hat{\mathbf{a}}$.*]

At time $t = 0$, particle 1 has its spin parallel to \mathbf{a} , while the other particle's spin is antiparallel to \mathbf{a} . Find the time required for both spins to reverse their directions.

5. States $|s, \sigma\rangle$ are formed by combining the states $|1, \sigma_1\rangle, |1, \sigma_2\rangle$ of two subsystems, each having spin 1. Show that the states of the combined system with $s = 2, 0$ are symmetric under the interchange $\sigma_1 \leftrightarrow \sigma_2$, whereas those with $s = 1$ are antisymmetric.

Two identical spin 1 particles, whose centre of mass is at rest, have combined spin \mathbf{S} , relative orbital angular momentum \mathbf{L} and total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$. Let (j, ℓ, s) denote the quantum numbers corresponding to the operators $(\mathbf{J}^2, \mathbf{L}^2, \mathbf{S}^2)$ of the combined system. Show that $\ell + s$ must be even. If $j = 1$, what are the possible values of ℓ and s ?

6. Show that $[\mathbf{L}^2, \mathbf{X}] = i\hbar(\mathbf{X} \times \mathbf{L} - \mathbf{L} \times \mathbf{X})$ and that $\mathbf{L} \cdot \mathbf{X} = 0$, where $\mathbf{L}^2 := \mathbf{L} \cdot \mathbf{L}$ is the total orbital angular momentum operator. Use these results to show that

$$[\mathbf{L}^2, [\mathbf{L}^2, \mathbf{X}]] = 2\hbar^2(\mathbf{L}^2 \mathbf{X} + \mathbf{X} \mathbf{L}^2).$$

By considering the matrix elements of this equation, show that

$$((\beta - \beta')^2 - 2(\beta + \beta')) \langle \ell', m' | \mathbf{X} | \ell, m \rangle = 0$$

where $\beta = \ell(\ell + 1)$ and $\beta' = \ell'(\ell' + 1)$ and $|\ell, m\rangle, |\ell', m'\rangle$ are orbital angular momentum eigenstates.

When hydrogen is immersed in a bath of radiation, transitions between the electron energy levels $|n, \ell, m\rangle$ and $|n', \ell', m'\rangle$ proceed at a rate proportional to $|\langle n', \ell', m' | \mathbf{X} | n, \ell, m \rangle|^2$. Show that this transition rate vanishes unless $|\ell - \ell'| = 1$ or $\ell' = \ell = 0$. By considering the parity operator, show that the transition rate also vanishes if $\ell = \ell' = 0$. By constructing an appropriate commutator, show also that allowed transitions also have $|m' - m| \leq 1$. [*This is a special case of the Wigner-Ekart theorem.*]

7. Consider the isotropic harmonic oscillator in three dimensions, with potential $V(r) = V_0 + \frac{1}{2}m\omega^2 r^2$. Write down the allowed energy eigenvalues. What is the degeneracy of the n^{th} level?

In a simple model of the atomic nucleus, each nucleon (proton or neutron) moves in the harmonic potential $V(r)$ created by the other nuclei. Explain why the nuclear isotopes ${}^4\text{He}$, ${}^{16}\text{O}$ and ${}^{40}\text{Ca}$ are especially stable. [*Helium, oxygen and calcium have atomic numbers $Z = 2, 8$ and 20 , respectively.*]

In the next two questions, you should assume that total angular momentum and parity are each conserved in the decay process.

8. Show that a particle of spin 1 cannot decay into two identical particles of spin 0. The ρ -meson has spin-1 and can decay into two spinless π -mesons (pions) with different electric charges. If the intrinsic parity of any pion is negative, what is the intrinsic parity of the ρ -meson?
9. A particle X is observed to undergo the decays $X \rightarrow \rho^+ + \pi^+$ and $X \rightarrow K + K$, where the kaon K has spin-0. What is the lowest value of the spin of X that is consistent with this, and what is the corresponding intrinsic parity of X ?