D18b **Principles of Quantum Mechanics: Sheet 2** Michaelmas 2016

1. Starting from the appropriate equation of motion for either operators or states, verify that
$$i\hbar \frac{\mathrm{d}}{\mathrm{d}t}\langle\phi|A|\psi\rangle = \langle\phi|[A,H]|\psi\rangle$$
holds in both the Heisenberg picture and the Schrödinger picture.

2. (a) Find and solve the Heisenberg equations of motion for the position $\hat{x}(t)$ and momentum $\hat{p}(t)$ of the harmonic oscillator. Write the solution in two forms, one involving $\hat{x}(0)$ and $\hat{p}(0)$ and the other involving the corresponding annihilation and creation operators a and $a^\dagger$ (also defined at $t = 0$).
   Show that for any complex number α there is a normalized state $|\psi_\alpha\rangle$ such that
   $$\langle\psi_\alpha|\hat{x}(t)|\psi_\alpha\rangle = \alpha e^{-i\omega t} + \alpha^* e^{i\omega t} ,$$
   the general solution for the *classical* oscillator. [Use the results of example 6 on sheet 1.]

   (b) The Hamiltonian for a point particle of mass m and charge e in a constant electric field $\mathbf{E}$ is
   $$H(\hat{\mathbf{x}},\hat{\mathbf{p}}) \;=\; \frac{1}{2m}\hat{\mathbf{p}}^2 \;-\; e\,\mathbf{E}\cdot\hat{\mathbf{x}} \;.$$
   Solve the Heisenberg equations of motion for $\hat{\mathbf{x}}(t)$ and $\hat{\mathbf{p}}(t)$, expressing your answer in terms of $\hat{\mathbf{x}}(0)$ and $\hat{\mathbf{p}}(0)$. Verify that $H(\hat{\mathbf{x}}(t),\hat{\mathbf{p}}(t)) = H(\hat{\mathbf{x}}(0),\hat{\mathbf{p}}(0))$.

3. Define annihilation, creation operators a_i, $a_i^\dagger$ for the three-dimensional harmonic oscillator with Hamiltonian
$$H = H_1 + H_2 + H_3 \qquad \text{where} \qquad H_i = \frac{\hat{p}_i^2}{2m} + \tfrac{1}{2}\, m\omega^2 \hat{x}_i^2 \qquad (i = 1,2,3).$$
Use these to find the energy levels and the space of states on which H acts. Assuming no internal structure, show that a complete set of commuting operators can be taken to be $\{a_1^\dagger a_1, a_2^\dagger a_2, a_3^\dagger a_3\}$. Show that the degeneracy of the n-th excited state is $\frac{1}{2}(n+1)(n+2)$.

4. Two quantum systems have spaces of states V and V'. Each space is two-dimensional with an orthonormal basis for V given by $\{|1\rangle, |2\rangle\}$ and for V' by $\{|1'\rangle, |2'\rangle\}$. An operator A is defined on V by $A|1\rangle = |2\rangle$, $A|2\rangle = |1\rangle$ and a similar operator A' is defined on V' by $A'|1'\rangle = |2'\rangle$, $A'|2'\rangle = |1'\rangle$. The identity operators on V and V' are I and I' respectively.

   (a) Consider the combined system whose space of states is represented by the tensor product $V \otimes V'$. With respect to a convenient basis, determine the matrix forms of the operators
   $$S = I \otimes I' \;+\; \lambda\, A \otimes A' , \qquad T = A \otimes I' \;+\; \mu\, I \otimes A' ,$$
   where λ, μ are real numbers. Find the eigenvalues of S.

(b) Show that $T^2 \propto S$ if λ and μ are suitably related. Hence, or otherwise, find the eigenvalues of T.

(c) Now suppose that V and V' are spaces of states for two indistinguishable fermions. What are the implications for the eigenvalues of S?

5. (a) The trace of an operator A acting on a vector space V is defined as $\mathrm{Tr}_V(A) = \sum_n \langle n|A|n\rangle$ where $\{|n\rangle\}$ is any orthonormal basis for V. If $a, a^\dagger$ are annihilation, creation operators and V is the usual space of states for an oscillator, show that $\mathrm{Tr}(x^{a^\dagger a}) = (1-x)^{-1}$ for $|x| < 1$.

(b) Suppose $V = V_1 \otimes V_2$ and $A = A_1 \otimes A_2$ where A_1, A_2 act on V_1, V_2 respectively. Show that $\mathrm{Tr}_V(A) = \mathrm{Tr}_{V_1}(A_1)\mathrm{Tr}_{V_2}(A_2)$.

*(c) Let $a_i, a_i{}^\dagger$ be a set of D annihilation, creation operators obeying $[a_i, a_j{}^\dagger] = \delta_{ij}$, $[a_i, a_j] = 0$, $[a_i^\dagger, a_j{}^\dagger] = 0$. Show that

$$\mathrm{Tr}(x^{\mathcal{N}}) = (1-x)^{-D} \qquad \text{where} \qquad \mathcal{N} = \sum_{i=1}^{D} a_i{}^\dagger a_i \ .$$

Find the degeneracy d_n of the eigenvalue n of the operator $\mathcal{N}$, by first explaining why

$$\mathrm{Tr}(x^{\mathcal{N}}) = \sum_{n\geq 0} d_n x^n.$$

6. (a) Use the angular momentum commutation relations to verify that

$$[\, J_3,\ J_1{}^2\,] = -[\, J_3,\ J_2{}^2\,] = i\hbar(J_1J_2 + J_2J_1) \text{ and } [\, J_3,\ J_1J_2 + J_2J_1\,] = 2i\hbar(J_2{}^2 - J_1{}^2)\ .$$

(b) Show that if $|\psi\rangle = |j\,m\rangle$, a simultaneous eigenstate of $\mathbf{J}^2$ and J_3, then $\langle A\rangle_\psi = \langle\psi|A|\psi\rangle = 0$ for any operator of the form $A = [B, J_3]$. Deduce that

$$\langle J_1\rangle_\psi = \langle J_2\rangle_\psi = 0 \qquad \text{and} \qquad \langle J_1{}^2\rangle_\psi = \langle J_2{}^2\rangle_\psi = \tfrac{1}{2}\hbar^2(\, j(j{+}1) - m^2\,)\ .$$

(c) Re-derive the expectation values in part (b) by writing J_1 and J_2 in terms of $J_\pm$.

7. A quantum system is in an angular momentum state $|\psi\rangle = |j\,m\rangle$, as in example 1, when a measurement is made of the angular momentum component $K = \mathbf{n}\cdot\mathbf{J}$, where the unit vector $\mathbf{n}$ lies in the x-z plane and makes an angle θ with the z-axis.

For $j = \frac{1}{2}$, find the probabilities that the measurement gives the results $\pm\frac{1}{2}\hbar$ by first evaluating $\langle K\rangle_\psi$. Show that in this case each probability is $\cos^2\frac{1}{2}\theta$ or $\sin^2\frac{1}{2}\theta$, depending on the value of m.

For $j = 1$, consider the expectation values of K and K^2 and hence determine the probabilities for the measurement to yield each of the results $\hbar$, 0, $-\hbar$.

8. Show, *without* using explicit forms for the Pauli matrices, that the relations

$$[\,\sigma_i,\ \sigma_j\,] \;=\; 2i\,\epsilon_{ijk}\sigma_k \qquad \text{and} \qquad (\mathbf{n}\cdot\sigma)^2 \;=\; 1\,,$$

where $\mathbf{n}$ is any unit vector, imply

$$\mathbf{a}\cdot\sigma\,\mathbf{b}\cdot\sigma \;=\; \mathbf{a}\cdot\mathbf{b} \,+\, i\,\mathbf{a}\times\mathbf{b}\cdot\sigma\ ,$$

for any vectors $\mathbf{a}$ and $\mathbf{b}$. Show also that

$$\exp(i\phi\,\mathbf{n}\cdot\sigma) \;=\; \cos\phi \;+\; i\mathbf{n}\cdot\sigma\,\sin\phi\ .$$

Hence deduce the results

$$\mathbf{n}\cdot\sigma\,\mathbf{a}\cdot\sigma\,\mathbf{n}\cdot\sigma \;=\; 2\,\mathbf{n}\cdot\mathbf{a}\,\mathbf{n}\cdot\sigma \;-\; \mathbf{a}\cdot\sigma\,,$$

$$\exp(\tfrac{1}{2}i\theta\mathbf{n}\cdot\sigma)\,\sigma\exp(-\tfrac{1}{2}i\theta\mathbf{n}\cdot\sigma) \;=\; \sigma\cos\theta \;+\; \mathbf{n}\,\mathbf{n}\cdot\sigma(1-\cos\theta) \;+\; \mathbf{n}\times\sigma\sin\theta\ .$$

9. A particle of spin $\frac{1}{2}$, with $\mathbf{S} = \frac{1}{2}\hbar\sigma$, is at rest (its spatial degrees of freedom may be ignored) and interacts with a uniform homogeneous magnetic field $B\mathbf{n}$ through its magnetic moment $\mu = \gamma\,\mathbf{S}$, the Hamiltonian being

$$H = -B\,\mathbf{n}\cdot\mu\ .$$

Use the results given in example 8 to find an expression for the time evolution operator $\exp(-itH/\hbar)$. Hence show that if the initial state of the particle is $|\chi\rangle$ then the probability of measuring it to be in an orthogonal state $|\chi'\rangle$ after a time t is

$$p(t) \;=\; |\,\langle\chi'|\mathbf{n}\cdot\sigma|\chi\rangle\,|^2\sin^2\tfrac{1}{2}\omega t \qquad \text{where} \qquad \omega = \gamma B\ .$$

Show also that in the Heisenberg picture the spin operators at time t are

$$\mathbf{S}(t) \;=\; \mathbf{S}\cos\omega t \;+\; \mathbf{n}\,\mathbf{n}\cdot\mathbf{S}\,(1-\cos\omega t) \;-\; \mathbf{n}\times\mathbf{S}\sin\omega t\,,$$

where $\mathbf{S}(0) = \mathbf{S}$, the Schrödinger picture operators (the pictures are defined to coincide at $t = 0$).

Simplify the expressions for $p(t)$ and $\mathbf{S}(t)$ in the special case where the magnetic field points along the y-axis and $|\chi\rangle$ and $|\chi'\rangle$ are eigenstates of spin up and spin down along the z-axis. Are these simplified expressions compatible with the results of examples 7 and 8, above?

10. Consider the addition of angular momentum 3 to angular momentum 1. Find the composition of combined states $|J\ M\rangle = \{|4\ 4\rangle,\ |4\ 3\rangle,\ |3\ 3\rangle,\ |3\ 2\rangle,\ |2\ 2\rangle,\ |2\ 1\rangle\}$ in terms of the initial angular momentum states. Try to explicitly calculate $|1\ 1\rangle$, finding the reason for the fact that it is zero. [You may set $\hbar = 1$ and assume the result $J_-|j\,m\rangle = \{\,(j{+}m)(j{-}m{+}1)\,\}^{\frac{1}{2}}\,|j\,m{-}1\rangle$.]