D18c **Principles of Quantum Mechanics: Sheet 3** Michaelmas 2016

1. A particle of mass m is confined to a three-dimensional box with sides $a < b < c$ (the wavefunction for the particle vanishes on the boundary of the box). Show that, with a suitable choice of coordinates, the allowed eigenfunctions and energy levels are

$$\psi_{npq}(x,y,z) = \Big(\frac{8}{abc}\Big)^{\frac{1}{2}} \sin\frac{n\pi x}{a}\sin\frac{p\pi y}{b}\sin\frac{q\pi z}{c}\ , \qquad E_{npq} = \frac{\hbar^2\pi^2}{2m}\Big(\frac{n^2}{a^2}+\frac{p^2}{b^2}+\frac{q^2}{c^2}\Big)$$

where $n,p,q = 1,2,\ldots$.

What are the two lowest energy levels, their degeneracies and associated wave functions, for two identical non-interacting spin-0 particles confined to the box? What are the corresponding results for two non-interacting, identical, spin-$\frac{1}{2}$ particles?

2. Show that, in the expression for states $|SM\rangle$ of spin S formed from the product of two spin-1 states, $|1\,m_1\rangle|1\,m_2\rangle$, the states with $S = 0$ and $S = 2$ are symmetric under the interchange of m_1 and m_2, whereas those with $S = 1$ are antisymmetric.

Two identical spin-1 particles, whose centre of mass is at rest, have combined spin $\mathbf{S}$, relative orbital angular momentum $\mathbf{L}$ and total angular momentum $\mathbf{J} = \mathbf{L} + \mathbf{S}$, with corresponding quantum numbers S, L and J. Show that $L + S$ must be even. If $J = 1$, what are the possible values of L and S?

3. (a) Let $\mathbf{J} = (J_1, J_2, J_3)$ and $|j\,m\rangle$ denote the standard angular momentum operators and states, so that, using units in which $\hbar = 1$,

$$\mathbf{J}^2\,|j\,m\rangle = j(j+1)|j\,m\rangle, \qquad J_3|j\,m\rangle = m|j\,m\rangle.$$

Show that $U(\theta) = \exp(-i\theta J_2)$ is unitary and define

$$J_i(\theta) = U(\theta)\,J_i\,U(\theta)^{-1} \quad \text{for} \quad i = 1,3\ .$$

Using the commutation relations for angular momentum show that

$$\frac{d^2 J_i(\theta)}{d\theta^2} + J_i(\theta) \;=\; 0 \quad \text{for} \quad i = 1,3\ .$$

Hence show that

$$J_1(\theta) = J_1\cos\theta - J_3\sin\theta\ , \qquad J_3(\theta) = J_1\sin\theta + J_3\cos\theta\ .$$

Deduce that $U(\frac{\pi}{2})\,|j\,m\rangle$ are eigenstates of J_1.

(b) For $j = \frac{1}{2}$, use the Pauli representation of operators and states to show that

$$U(\theta)|\!\uparrow\rangle \;=\; \cos\tfrac{1}{2}\theta\,|\!\uparrow\rangle \;+\; \sin\tfrac{1}{2}\theta\,|\!\downarrow\rangle\ , \qquad U(\theta)|\!\downarrow\rangle \;=\; -\sin\tfrac{1}{2}\theta\,|\!\uparrow\rangle \;+\; \cos\tfrac{1}{2}\theta\,|\!\downarrow\rangle$$

where $|\!\uparrow\rangle = |\frac{1}{2}\ \frac{1}{2}\rangle$ and $|\!\downarrow\rangle = |\frac{1}{2}\ -\frac{1}{2}\rangle$. Verify in this representation that $U(\frac{\pi}{2})|\!\uparrow\rangle$ and $U(\frac{\pi}{2})|\!\downarrow\rangle$ are eigenstates of J_1.

(c) Show that for two spin-$\frac{1}{2}$ particles the composite state

$$\frac{1}{\sqrt{2}}\big(\,|\!\uparrow\rangle|\!\downarrow\rangle \;-\; |\!\downarrow\rangle|\!\uparrow\rangle\,\big)$$

is unchanged by a transformation $|\!\uparrow\rangle \mapsto U(\theta)|\!\uparrow\rangle$ and $|\!\downarrow\rangle \mapsto U(\theta)|\!\downarrow\rangle$ applied to all one-particle states. How does this relate to the angular momentum properties of the two-particle state?

4. What is the unitary operator $U(\alpha)$ corresponding to translation through α for a one-dimensional quantum system with position $\hat{x}$ and momentum $\hat{p}$? Calculate $[\hat{x}, U(\alpha)]$ and show that the result is consistent with the assumption that position eigenstates obey $|x{+}\alpha\rangle = U(\alpha)|x\rangle$. Given this assumption, express the wavefunction for $U(\alpha)|\psi\rangle$ in terms of the wavefunction $\psi(x)$ for $|\psi\rangle$.

   If the system is a one-dimensional harmonic oscillator of mass m and frequency ω, show that

$$U(\alpha) \;=\; e^{-\frac{1}{2}\gamma^2} e^{\gamma a^\dagger} e^{-\gamma a} \qquad \text{where} \qquad \gamma = \alpha(m\omega/2\hbar)^{\frac{1}{2}} \;.$$

   Deduce that if $\psi_n(x)$ are wavefunctions for the usual normalised states with energies $\hbar\omega(n{+}\frac{1}{2})$, then

$$\psi_0(x{-}\alpha) \;=\; e^{-\frac{1}{2}\gamma^2} \sum_{n=0}^{\infty} \frac{1}{\sqrt{n!}}\, \gamma^n \psi_n(x) \;.$$

   [Recall that $[A, e^B] = [A,B]e^B$ and $e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$ provided $[A,B]$ commutes with A and B.]

5. Write down the commutation relations for the components of a vector operator $\mathbf{V} = (V_1, V_2, V_3)$ and the angular momentum operator $\mathbf{J} = (J_1, J_2, J_3)$. Use these to show that

$$\mathbf{V}(\theta) = \exp(i\theta \mathbf{n}{\cdot}\mathbf{J}/\hbar)\, \mathbf{V}\, \exp(-i\theta\mathbf{n}{\cdot}\mathbf{J}/\hbar)$$

   satisfies

$$\mathbf{V}'(\theta) = \mathbf{n}{\times}\mathbf{V}(\theta)$$

   where $\mathbf{n}$ is a unit vector and θ a real parameter. Deduce that $\mathbf{n}{\cdot}\mathbf{V}(\theta) = \mathbf{n}{\cdot}\mathbf{V}$ and hence that

$$\mathbf{V}''(\theta) + \mathbf{V}(\theta) = (\mathbf{n}{\cdot}\mathbf{V})\,\mathbf{n}\,.$$

   Solve this equation to find $\mathbf{V}(\theta)$ in terms of $\mathbf{V}$ and interpret your result.

6. Show that a particle of spin 1 cannot decay into two identical particles of spin 0. The ρ-meson has spin 1 and can decay into two spinless π-mesons, or pions, with different charges. If the intrinsic parity of any π is negative, what is the intrinsic parity of the ρ?

7. A particle X is observed to undergo the decays $X \to \rho^+ + \pi^-$ and $X \to K + K$, where K is a particle of spin 0. What is the lowest value for the spin of X that is consistent with this, and what is the corresponding intrinsic parity of X? [Assume that total angular momentum and parity are conserved is all these processes.]

8. A harmonic oscillator of mass m, frequency ω and charge e is perturbed by a constant electric field of strength $\mathcal{E}$, resulting in a new term $H' = -e\mathcal{E}\hat{x}$ in the Hamiltonian. Calculate the change in the energy levels to order $\mathcal{E}^2$ and compare with the exact result.

9. A particle of spin $\frac{1}{2}$ is at rest in a magnetic field B parallel to the z-axis. A small additional magnetic field B' is then switched on parallel to the x-axis, so that the Hamiltonian becomes

$$H = -\tfrac{1}{2}\hbar\gamma(B\sigma_3 + B'\sigma_1)$$

   (where σ_i are the Pauli matrices). Starting from the energy levels and eigenstates when $B' = 0$, use perturbation theory to calculate the corrections to the energies to order B'^2 and compare with the exact answer.