

Principles of Quantum Mechanics - Problems 3

Please *email me* with any comments about these problems, particularly if you spot an error.

1. Consider a $d = 1$ harmonic oscillator of mass m and frequency ω , with raising and lowering operators A^\dagger and A . Show that the translation operator $U(c)$ may be written as

$$U(c) = e^{-\frac{1}{2}\gamma^2} e^{-i\gamma A^\dagger} e^{-i\gamma A} \quad \text{where} \quad \gamma := c\sqrt{\frac{m\omega}{2\hbar}}.$$

Deduce that if $\psi_n(x)$ are the normalised position space wavefunctions for states with energies $\hbar\omega(n + \frac{1}{2})$, then

$$\psi_0(x - c) = e^{-\frac{1}{2}\gamma^2} \sum_{n=0}^{\infty} \frac{(-i\gamma)^n}{\sqrt{n!}} \psi_n(x).$$

[Recall that $[A, e^B] = [A, B]e^B$ and $e^A e^B = e^{A+B} e^{\frac{1}{2}[A, B]}$ provided $[A, B]$ commutes with A and B .]

2. A Fermi oscillator has Hamiltonian $H = B^\dagger B$ where

$$B^\dagger B + BB^\dagger = 1 \quad (\text{the anti-commutator})$$

and $B^2 = 0$. Find the eigenvalues of H . If $|0\rangle$ is a state obeying $H|0\rangle = 0$ and $\langle 0|0\rangle = 1$, find $B|0\rangle$ and $B^\dagger|0\rangle$. Explain the connection between the spectrum of $B^\dagger B$ and the Pauli exclusion principle.

3. Consider an free particle with mass m . Show that energy eigenstates with definite total angular momentum obey $H_\ell|E, \ell\rangle = E|E, \ell\rangle$ where

$$H_\ell := \frac{1}{2m} \left(P_r^2 + \frac{\ell(\ell+1)\hbar^2}{|\mathbf{X}|^2} \right)$$

and $P_r := (\hat{\mathbf{X}} \cdot \mathbf{P} + \mathbf{P} \cdot \hat{\mathbf{X}})/2$. Now let

$$A_\ell := \frac{1}{\sqrt{2m}} \left(iP_r - \frac{(\ell+1)\hbar}{|\mathbf{X}|} \right)$$

Find $[A_\ell, A_\ell^\dagger]$ in terms of the H_ℓ s. What is the state $A_\ell|E, \ell\rangle$? Show that for $E > 0$ there is no upper bound on ℓ and interpret this result physically.

4. A $d = 1$ harmonic oscillator of mass m and frequency ω is perturbed by $\Delta = \lambda X^4$. By expressing the position operator X in terms of the raising and lowering operators A^\dagger & A , show that the first order change in energy of the n^{th} excited state is

$$\delta E_n = 3\lambda(2n^2 + 2n + 1) \left(\frac{\hbar}{2m\omega} \right)^2.$$

What is the radius of convergence in λ of perturbation theory here?

5. A particle of mass m is confined by the infinite square well potential

$$V(x) = \begin{cases} 0 & \text{when } |x| < a \\ \infty & \text{when } |x| > a. \end{cases}$$

Show that, to first order in λ , the energy levels of this particle are unchanged when the potential is perturbed by $\delta V(x) = \lambda x/a$. Show that the ground-state wavefunction is changed, becoming

$$\psi_\lambda(x) = \frac{1}{\sqrt{a}} \cos(\pi x/2a) + \frac{16\lambda}{\pi^2 E_1 \sqrt{a}} \sum_{n=2,4,\dots} (-1)^{n/2} \frac{n}{(n^2-1)^3} \sin(n\pi x/2a) + O(\lambda^2)$$

where E_1 is the ground state energy. Explain why this wavefunction does not have well-defined parity. Will the second order correction to the ground-state energy be positive or negative? [*Detailed calculation is not required.*]

Suppose now that the initial well is only finitely deep. Will perturbation theory converge? Comment on the relevance of this to the Stark effect in hydrogen.

6. An atomic nucleus has finite size and inside the electrostatic potential $\Phi(r)$ deviates from Ze/r . Take the proton's radius to be $a_p \approx 10^{-15}$ m and its charge density to be uniform. Treating the difference between $\Phi(r)$ and Ze/r as a perturbation of the gross structure Hamiltonian of the hydrogen, calculate the first order change in the energy of the ground state of hydrogen. [*Use the fact that $a_p \ll a_0$ (the Bohr radius) to approximate the integral you encounter.*]

Why is the change in energy of any state with $\ell \geq 1$ extremely small?

7. An isotropic harmonic oscillator in $d = 2$ has mass m and frequency ω . Write down a basis of its energy eigenstates. What is the energy and degeneracy of the n^{th} excited state?

This oscillator is perturbed by small potential $\delta V = \lambda xy$. Show that this perturbation lifts the degeneracy of the first excited state, producing states with energies $2\hbar\omega \pm \lambda\hbar/2m\omega$. What are the corresponding eigenstates? Which states remain degenerate in the presence of this perturbation? What is the radius of convergence (in λ) of this perturbation series?