Copyright © 2023 University of Cambridge. Not to be quoted or reproduced without permission.

Applications of Quantum Mechanics: Example Sheet 1

Alejandra Castro, January 2024

1. A particle of mass m is scattered by the symmetric square well potential given by

$$V(x) \;=\; \begin{cases} -V_0 & |x| < a/2 \\ 0 & |x| > a/2 \end{cases} \qquad \text{with } V_0 > 0$$

The incoming and outgoing plane waves of even $(+)$ and odd $(-)$ parity are

$$\begin{aligned} \mathcal{I}_+(k;x) &= e^{-ik|x|} \quad , \quad \mathcal{I}_-(k;x) = \text{sign}(x)\, e^{-ik|x|} \\ \mathcal{O}_+(k;x) &= e^{+ik|x|} \quad , \quad \mathcal{O}_-(k;x) = -\text{sign}(x)\, e^{+ik|x|} \end{aligned}$$

The corresponding scattering states for $|x| > a/2$ are

$$\begin{aligned} \psi_+(k;x) &= \mathcal{I}_+(k;x) + S_{++}(k)\mathcal{O}_+(k;x) \\ \psi_-(k;x) &= \mathcal{I}_-(k;x) + S_{--}(k)\mathcal{O}_-(k;x) \end{aligned}$$

where S_{++} and S_{--} are the diagonal elements of the S-matrix in the parity basis. By imposing the boundary condition that ψ'/ψ is continuous at $x = a/2$, show that

$$S_{++} = -e^{-ika}\,\frac{q\tan(qa/2) - ik}{q\tan(qa/2) + ik} \quad , \quad S_{--} = e^{-ika}\frac{q + ik\tan(qa/2)}{q - ik\tan(qa/2)}$$

where $q^2 = k^2 + U_0$ and $U_0 = 2mV_0/\hbar^2$. Interpret the poles and zeros of S_{++} and S_{--} in terms of bound states.

2. Carry out a similar analysis to that in Question 1, this time for the potential

$$V(x) = V_0\Big[\delta(x-1) + \delta(x+1)\Big]$$

with $V_0 > 0$. Interpret the poles and zeros of S_{++} and S_{--} in the complex k-plane as resonances, in the case where $V_0 \gg 1$. Show that, approximately, the pole position in S_{++} with the smallest real part lies at

$$k = \frac{\pi}{2} - \frac{\pi}{2U_0} + \frac{\pi}{2U_0^2} - i\frac{\pi^2}{4U_0^2}$$

where $U_0 = 2mV_0/\hbar^2$.

3. A particle of mass m and energy $\hbar^2 k^2/2m$ scatters off a hard sphere of radius a. Show that the phase shifts δ_l obey

$$\tan \delta_l = \frac{j_l(ka)}{n_l(ka)}$$

4. A particle of mass m and energy $\hbar^2 k^2/2m$ scatters off the attractive potential

$$V(r) = \begin{cases} -\hbar^2\gamma^2/2m & \text{if } r < a \\ 0 & \text{if } r \geq a \end{cases}$$

Show that the phase shifts obey

$$\tan \delta_l = \frac{q\, j_l(ka) j_l'(qa) - k\, j_l'(ka) j_l(qa)}{q\, n_l(ka) j_l'(qa) - k\, n_l'(ka) j_l(qa)}$$

where $q^2 = k^2 + \gamma^2$. Hence show that, as $k \to 0$, $\delta_l \sim (ka)^{2l+1}$.

[*Hint:* You will require the small x behaviours of $j_l(x)$ and $n_l(x)$ given in the handout. To answer the first part, consider the requirements of regularity of the wavefunction at $r = 0$.]

5. Consider a particle of essentially zero energy scattering off a repulsive spherical potential

$$V(r) = \begin{cases} +\hbar^2\gamma^2/2m & \text{if } r < a \\ 0 & \text{if } r \geq a \end{cases}$$

By considering the s-wave wavefunction $r\psi(r)$ for $r \geq a$, show that the scattering length is positive.

Consider now the attractive potential from Question 4. Show that for a sufficiently small the scattering length is negative. Show further that as a increases to a critical value a_0, the scattering length becomes infinite. What occurs for $a > a_0$?

Copyright © 2023 University of Cambridge. Not to be quoted or reproduced without permission.

6. Consider the spherical shell of delta-functions,

$$V(r) = \frac{\hbar^2 \lambda}{2m} \delta(r-a)$$

Show that the phase shifts are given by

$$\tan \delta_l = -\frac{k\lambda a^2 [j_l(ka)]^2}{1 - k\lambda a^2 j_l(ka) n_l(ka)}$$

[*Hint*: The δ-function leads to two continuity conditions at $r = a$. Either take appropriate ratios of these conditions, or arrange them as two linear equations for the unknown amplitudes and recognise that the determinant of coefficients must vanish. You will need the Wronskian condition that $x^2(j_l(x)n_l'(x) - j_l'(x)n_l(x)) = 1$.]

For λ large, show that δ_l is close to the result given in Question 3 *unless* $k \simeq k_0$ where $j_l(k_0 a) = 0$. In the latter case, show that we can write $\tan \delta_l \approx -A/(k - k_0')$ where $k_0' = k_0 + \mathcal{O}(\lambda^{-1})$ and $A = \mathcal{O}(\lambda^{-2})$. What is the interpretation of this result?

7. The s-wave wavefunction $\psi_k(r) = \chi_k(r)/r$ obeys the Schrödinger equation

$$-\frac{d^2 \chi_k}{dr^2} + V(r)\chi_k = k^2 \chi_k \,,$$

with $k \in \mathbf{C}$ and where $\chi_k(r)$ satisfies the boundary conditions $\chi_k(0) = 0$ and $\chi_k'(0) = 1$. Show that

$$\chi_k(r) = \chi_{-k}(r) \quad \text{and} \quad [\chi_k(r)]^* = \chi_{k^*}(r)$$

For large r

$$\chi_k(r) \sim \frac{i}{2k} \left[f(k)\, e^{ikr} - f(-k)\, e^{-ikr} \right] \,.$$

Identify the S-matrix element $S(k)$ for scattering in the $l = 0$ sector in terms of $f(k)$. What conditions do $S(k)$ obey?

Copyright © 2023 University of Cambridge. Not to be quoted or reproduced without permission.

8. Resonance scattering in the $l = 0$ sector is modelled by an S-matrix

$$S(k) = \frac{k - K_0 - i\gamma}{k - K_0 + i\gamma}$$

with $K_0 \gg \gamma > 0$. For k real, $S(k) = e^{2i\delta_0(k)}$. Show that

$$\tan\delta_0(k) = \frac{\gamma}{K_0 - k} \quad \text{and} \quad \sigma \simeq \frac{4\pi\gamma^2}{K_0^2}\frac{1}{(K_0 - k)^2 + \gamma^2}.$$

Show that for $k = K_0 - i\gamma$, the magnitude of the complex wavefunction has an exponential growth for large r at fixed t and, including the time-dependent factor $e^{-iEt/\hbar}$, that it decays with time at fixed r.

9. Calculate, in the Born approximation, the differential cross-section $d\sigma/d\Omega$ for a particle of mass m scattering off the potential

$$V(r) = \frac{Ae^{-\mu r}}{r}$$

as a function of the momentum transfer. Express this as a function of energy E and scattering angle θ, and show that, for large E, $d\sigma/d\Omega$ is proportional to E^{-2} at fixed $\theta \neq 0$. Show that $d\sigma/d\Omega$ is independent of E at $\theta = 0$. Show that the total cross section σ is proportional to E^{-1} for large E.

10. Calculate the Born approximation to the differential cross-section for the following potentials:

$$-V_0\, e^{-\lambda^2 r^2} \quad , \quad \frac{V_0}{r^2} \quad , \quad V_0\,\delta(r-a) \quad , \quad V(r) = \begin{cases} V_0, & r < a \\ 0\,, & r > a\,. \end{cases}$$

[*Note*: $\int_0^\infty (\sin x/x)\, dx = \pi/2$.]

Copyright © 2023 University of Cambridge. Not to be quoted or reproduced without permission.