Mathematical Tripos Part II Anthony Challinor

Electrodynamics Michaelmas 2025

Problems 3.

SI units are used.

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1. A dielectric sphere of radius a and relative permittivity ϵ_r is placed at rest at the origin in a uniform electric field $\mathbf{E}_0$. Show that the resulting electric field is uniform inside the sphere, with

$$\mathbf{E}_{\text{int}} = \frac{3}{2+\epsilon_r}\mathbf{E}_0 \,.$$

Show further that the field outside the sphere is the superposition of $\mathbf{E}_0$ and the field of an electric dipole with moment

$$\mathbf{p} = 4\pi\epsilon_0 \left(\frac{\epsilon_r - 1}{\epsilon_r + 2}\right) a^3 \mathbf{E}_0 \,.$$

By considering the macroscopic polarisation $\mathbf{P}$ inside the sphere, calculate the bound surface charge density σ. Verify explicitly that the electric dipole moment of σ is $\mathbf{p}$.

2. A permanent bar magnet can be modelled as a long cylinder of length l and radius a with uniform macroscopic magnetisation $\mathbf{M}$ along its axis. By considering the bound surface current due to $\mathbf{M}$, or otherwise, show that for $a \ll l$ the magnetic field on axis at the centre of the cylinder is $\mathbf{B} = \mu_0\mathbf{M}$. Show further that the magnetic field on axis at the end of the cylinder is $\mathbf{B} = \mu_0\mathbf{M}/2$.

Sketch the field lines of the magnetic field $\mathbf{B}$ and the magnetising field $\mathbf{H}$ both inside and outside the cylinder.

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3. Two plane, semi-infinite dielectric slabs with refractive index n are separated by a vacuum region of width d. A plane wave of wavenumber k is incident on the vacuum region from one of the slabs at an angle of incidence θ_I to the normal, with $n \sin\theta_I > 1$. The electric field of the incident radiation is normal to the plane of incidence (normal polarization). Show that some of the radiation is transmitted across the gap, and that the ratio of the amplitudes of the electric fields of the transmitted and incident waves satisfies

$$\left|\frac{E_T}{E_I}\right|^2 = \frac{1}{1+\alpha^2 \sinh^2 \kappa d},$$

where

$$\alpha = \frac{k(1-1/n^2)}{2\kappa \cos\theta_I}$$

and $\kappa = k\sqrt{\sin^2\theta_I - 1/n^2}$. This process is known as *frustrated total-internal reflection* and is analogous to quantum tunnelling.

[*Hint: you will need to consider both growing and decaying evanescent waves in the gap since it is of finite width.*]

4. Consider a Gaussian wavepacket of the form

$$\mathbf{E}(t,z) = \Re \int_{-\infty}^{\infty} \frac{dk}{\sqrt{2\pi}} \mathbf{E}_0(k) e^{i(kz-\omega(k)t)},$$

where $\mathbf{E}_0(k) \propto \exp[-(k-k_0)^2/(2\sigma^2)]$, propagating in a dispersive medium. Ignore any dissipative effects so that k and $\omega(k)$ are real-valued. Expanding the dispersion relation to second-order about k_0, i.e.,

$$\begin{aligned}\omega(k) &\approx \omega(k_0) + (k-k_0)\omega'(k_0) + \frac{1}{2}(k-k_0)^2\omega''(k_0) \\ &= k_0 v_{\mathrm{p}}(k_0) + (k-k_0)v_{\mathrm{g}}(k_0) + \frac{1}{2}(k-k_0)^2 v_{\mathrm{g}}'(k_0),\end{aligned}$$

where primes denote derivatives with respect to k, and v_{p} and v_{g} are the phase and group velocities at k_0, show that

$$\mathbf{E}(t,z) \propto \sqrt{\frac{\sigma}{\sigma_z(t)}} e^{-(z-v_{\mathrm{g}}t)^2/[2\sigma_z^2(t)]} \cos\left\{k_0(z-v_{\mathrm{p}}t) + \sigma^2 v_{\mathrm{g}}' t (z-v_{\mathrm{g}}t)^2/[2\sigma_z^2(t)] - \phi(t)/2\right\}.$$

Here, $\sigma_z^2(t) = 1/\sigma^2 + (\sigma v_{\mathrm{g}}' t)^2$ and $\tan\phi(t) = \sigma^2 v_{\mathrm{g}}' t$. Note that the Gaussian modulation of the wavepacket has a width $\sigma_z(t)$ that spreads in time for non-zero v_{g}'.

Show that while the fractional increase in the width of the wavepacket is small, the terms involving v_{g}' have a negligible effect on the phase of the oscillation for $|z - v_{\mathrm{g}}t| \leq \sigma_z(t)$.

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5. A conductor has a frequency-dependent conductivity $\sigma(\omega)$. Assume that the relative permittivity and permeability can be approximated as $\epsilon_r = 1$ and $\mu_r = 1$ at all frequencies. A plane electromagnetic wave of frequency ω is incident normally on such a semi-infinite conductor from vacuum. Show that the ratio of the amplitudes of the electric fields of the reflected and incident waves is

$$\frac{E_R(\omega)}{E_I(\omega)} = \frac{1 - \sqrt{\epsilon_r^{\text{eff}}(\omega)}}{1 + \sqrt{\epsilon_r^{\text{eff}}(\omega)}}\,,$$

where the effective relative permittivity $\epsilon_r^{\text{eff}}(\omega) = 1 + i\sigma(\omega)/(\epsilon_0\omega)$.

For the Drude model, $\sigma(\omega) = \sigma_{\text{DC}}/(1 - i\omega\tau)$, calculate the fraction $R(\omega)$ of the incident power that is reflected as a function of frequency. In particular, show the following.

(a) For $\omega\tau \ll 1$, $R(\omega) \approx 1 - 2\omega\delta/c$ to first-order in $\omega\delta/c$, where the skin depth $\delta = \sqrt{2/(\mu_0\omega\sigma_{\text{DC}})}$.

(b) For $\omega\tau \gg 1$, but $\omega < \omega_p$ where the plasma frequency $\omega_p = \sqrt{\sigma_{\text{DC}}/(\epsilon_0\tau)}$, all power is reflected $[R(\omega) = 1]$.

(c) For $\omega\tau \gg 1$ and $\omega \gg \omega_p$, $R(\omega) \approx [\omega_p/(2\omega)]^4$ to leading order in ω_p/ω.

6. (*Optional, for enthusiasts.*) Suppose that space is filled with a uniform, non-dispersive dielectric medium such that the refractive index is n. Write the macroscopic Maxwell's equations in terms of the magnetic vector potential $\mathbf{A}$ (such that $\mathbf{B} = \nabla \times \mathbf{A}$) and electric potential ϕ (with $\mathbf{E} = -\partial\mathbf{A}/\partial t - \nabla\phi$), to show that, with a suitable choice of gauge,

$$\nabla^2\mathbf{A} - \frac{n^2}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = -\mu_0\mathbf{J}_{\text{free}}$$
$$\nabla^2\phi - \frac{n^2}{c^2}\frac{\partial^2\phi}{\partial t^2} = -\frac{\rho_{\text{free}}}{\epsilon_0 n^2}\,.$$

Construct the Green's function for the wave operator in these equations to show that the (retarded) vector potential generated by the free current is

$$\mathbf{A}(t,\mathbf{x}) = \frac{\mu_0}{4\pi}\int \frac{\mathbf{J}_{\text{free}}(t - n|\mathbf{x}-\mathbf{x}'|/c, \mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\, d^3\mathbf{x}'\,.$$

A point charge q moves through the dielectric with velocity $\mathbf{v}$, so that $\mathbf{J}_{\text{free}}(t,\mathbf{x}) = q\mathbf{v}\delta^{(3)}(\mathbf{x} - \mathbf{v}t)$. Show that

$$\mathbf{A}(t,\mathbf{x}) = \frac{\mu_0 q}{4\pi}\mathbf{v}f(\mathbf{x} - \mathbf{v}t) \qquad \text{where} \qquad f(\mathbf{x}) = \int \frac{1}{|\mathbf{r}|}\delta^{(3)}(\mathbf{x} + \mathbf{r} + n\mathbf{v}|\mathbf{r}|/c)\, d^3\mathbf{r}\,.$$

(Note that the integration variable here is $\mathbf{r} = \mathbf{x}' - \mathbf{x}$.) Taking the velocity to be along the positive z-direction, $\mathbf{v} = \beta c\hat{\mathbf{z}}$, show that

$$f(\rho, z) = \int_{-\infty}^{\infty} \left(\rho^2 + r_\parallel^2\right)^{-1/2} \delta\left(z + r_\parallel + n\beta\sqrt{\rho^2 + r_\parallel^2}\right)\, dr_\parallel\,,$$

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where $\rho = \sqrt{x^2 + y^2}$. For the case $n\beta > 1$ (i.e., the speed of the particle exceeds the speed of light in the dielectric), show that $f(\rho, z)$ vanishes unless $z \leq -\rho\sqrt{n^2\beta^2 - 1}$. Show further that

$$\mathbf{A}(t, \mathbf{x}) = \mathbf{v}\frac{\mu_0 q}{4\pi}\frac{2}{\sqrt{(z - vt)^2 + \rho^2(1 - n^2\beta^2)}} \qquad \left(z - vt \leq -\rho\sqrt{n^2\beta^2 - 1}\right) ,$$

where $v = |\mathbf{v}|$. These shock-like fields describe *Cherenkov radiation.*

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