Mathematical Tripos Part II **Lent Term 2024**

General Relativity **Dr. J.M. Evans**

Example Sheet 2

1. (i) Consider a parametrised curve given by $x^\mu(\lambda)$ or $\tilde{x}^\alpha(\lambda)$, depending on the choice of coordinates. Check that $T^\mu = \mathrm{d}x^\mu/\mathrm{d}\lambda$ and $\tilde{T}^\alpha = \mathrm{d}\tilde{x}^\alpha/\mathrm{d}\lambda$ are related by the transformation rule for components of a vector.

Using, in addition, the relation between the metric components $g_{\mu\nu}$ and $\tilde{g}_{\alpha\beta}$ in the coordinates above, check that $T_\mu = g_{\mu\nu}T^\nu$ and $\tilde{T}_\alpha = \tilde{g}_{\alpha\beta}\tilde{T}^\beta$ are related by the transformation rule for components of a covector.

(ii) With notation as (i), start from the geodesic equation in coordinates x^μ,

$$\frac{\mathrm{d}T^\mu}{\mathrm{d}\lambda} + \Gamma^\mu_{\nu\,\rho}\,T^\nu\,T^\rho = 0\,,$$

substitute for T^μ in terms of $\tilde{T}^\alpha$, then compare with the geodesic equation in coordinates $\tilde{x}^\alpha$ to show that

$$\frac{\partial x^\mu}{\partial \tilde{x}^\alpha}\,\tilde{\Gamma}^\alpha_{\beta\,\gamma} = \frac{\partial x^\nu}{\partial \tilde{x}^\beta}\frac{\partial x^\rho}{\partial \tilde{x}^\gamma}\,\Gamma^\mu_{\nu\,\rho} + \frac{\partial^2 x^\mu}{\partial \tilde{x}^\beta \partial \tilde{x}^\gamma}\,.$$

2. Let x^μ be coordinates on a manifold of dimension n, and let $g_{\mu\nu}$ be components of a metric with a Taylor expansion about the origin:

$$g_{\mu\nu} = \eta_{\mu\nu} + C_{\mu\nu\rho}\,x^\rho + \frac{1}{2!}C'_{\mu\nu\rho\sigma}\,x^\rho x^\sigma + \ldots\,.$$

Consider a change of coordinates, also given by a Taylor expansion:

$$\tilde{x}^\alpha = x^\alpha + \frac{1}{2!}A^\alpha{}_{\beta\gamma}\,x^\beta x^\gamma + \frac{1}{3!}A'^{\,\alpha}{}_{\beta\gamma\delta}\,x^\beta x^\gamma x^\delta + \ldots\,.$$

What can be assumed about symmetries of the coefficients C, C', A and A', without loss of generality?

With these assumptions, show that C and A each have $\frac{1}{2}n^2(n+1)$ independent components. Find the number of independent components of C' and A' and hence show that the difference is $\frac{1}{12}n^2(n^2-1)$. What can be deduced from these observations?

3. A static spacetime has line element

$$\mathrm{d}s^2 = -e^{2\phi/c^2}c^2\,\mathrm{d}t^2 + g_{ij}\,\mathrm{d}x^i\mathrm{d}x^j$$

where ϕ and g_{ij} are functions only of x^i, with i, $j = 1, 2, 3$, while $x^0 = ct$. Show that

$$\Gamma^0_{0\,i} = \frac{1}{c^2}\frac{\partial\phi}{\partial x^i}$$

and find $\Gamma^0_{0\,0}$, $\Gamma^i_{0\,0}$ and $\Gamma^0_{i\,j}$, where $\Gamma^\alpha_{\beta\,\gamma}$ is the Levi-Civita connection (give your answer in terms of g^{ij}, the inverse of g_{ij}, where necessary).

An observer is at rest in the coordinate system above and has 4-velocity $V^\alpha = \mathrm{d}x^\alpha/\mathrm{d}\tau$, where τ is proper time. Use the normalisation condition $V^\alpha V_\alpha = -c^2$ to find V^0 and V_0.

The 4-acceleration is defined by $A^\alpha = (\mathrm{d}V^\alpha/\mathrm{d}\tau) + \Gamma^\alpha_{\beta\,\gamma}V^\beta V^\gamma$. Show that $A_\alpha = \partial\phi/\partial x^\alpha$.

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4. If A_α is a covector field, let

$$F_{\alpha\beta} = \frac{\partial A_\beta}{\partial x^\alpha} - \frac{\partial A_\alpha}{\partial x^\beta}\,.$$

By writing this expression in terms of covariant derivatives, or otherwise, show that $F_{\alpha\beta}$ is a tensor field.

If U^α and V^α are vector fields, the *commutator* $[\,U,\,V\,]^\alpha$ is defined by

$$[\,U,\,V\,]^\alpha = U^\beta \frac{\partial V^\alpha}{\partial x^\beta} - V^\beta \frac{\partial U^\alpha}{\partial x^\beta}\,.$$

Show that $[\,U,\,V\,]^\alpha$ is a vector field.

5. The metric $g_{\alpha\beta}(x)$ has the property that, if each point x^α is mapped to $\phi^\alpha(x)$, distances are unaltered. Such a mapping is called an *isometry* for this metric. Show that

$$g_{\alpha\beta}\big(\phi(x)\big)\frac{\partial\phi^\alpha}{\partial x^\gamma}\frac{\partial\phi^\beta}{\partial x^\delta} = g_{\gamma\delta}(x)\,.$$

Setting $\phi^\alpha(x) = x^\alpha + \epsilon\xi^\alpha(x)$, where ϵ is small, show that to first order in ϵ

$$\xi^\gamma\frac{\partial g_{\alpha\beta}}{\partial x^\gamma} + g_{\gamma\beta}\frac{\partial\xi^\gamma}{\partial x^\alpha} + g_{\gamma\alpha}\frac{\partial\xi^\gamma}{\partial x^\beta} = 0\,.$$

Show also, using the formula for the connection components, that this condition can be written in tensorial form as

$$\nabla_\alpha\xi_\beta + \nabla_\beta\xi_\alpha = 0\,.$$

This is known as *Killing's equation* and solutions ξ_α are called *Killing covector fields*.

(i) Consider a massive, freely falling particle moving along an affinely parametrized geodesic with tangent vector V^α. Show that Killing's equation implies that $\mathcal{E} = \xi_\alpha V^\alpha$ is constant along the geodesic.

(ii) If a Killing vector field takes the form $\xi^\alpha = (1, \mathbf{0})$ in a given coordinate system, show that $\partial g_{\alpha\beta}/\partial x^0 = 0$. What is the physical interpretation of $\mathcal{E}$ in this case?

6. Write down the radial Euler-Lagrange equation for the Schwarzschild metric and show that, in the case of a circular geodesic orbit in the equatorial plane, this determines the period of the orbit in terms of Schwarzschild coordinate time. How does this relate to the Newtonian result?

Alice orbits the Earth on a circular geodesic path of radius R, while Bob stays at home on the Earth's surface, at radius R_E (the Earth is taken to be a non-rotating sphere).

(i) Show that Alice's proper time τ_A is related to the Schwarzschild time coordinate t by

$$d\tau_A^2 = (1 - 3M/R)\,dt^2\,.$$

(ii) How is Bob's proper time τ_B related to the coordinate t? Deduce that Bob and Alice age at the same rate if $R = \frac{3}{2}R_E$.

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7. A massive particle moves on a circular orbit of radius R in Schwarzschild spacetime, with E and h denoting the usual first integrals for t and ϕ (take units with $G = c = 1$). Use the r Euler-Lagrange equation to find conditions that R must satisfy, and show that they are the same as would be obtained by considering motion in the 1-dimensional potential $V_{\text{eff}}(r)$, where

$$2V_{\text{eff}}(r) \;=\; \Big(1 + \frac{h^2}{r^2}\Big)\Big(1 - \frac{2M}{r}\Big) \;-\; 1\,.$$

(i) Express h in terms of R and M and deduce that there are circular orbits for $R > 3M$.

(ii) Show that these orbits are stable if $R > 6M$ and unstable if $3M < R < 6M$.

(iii) Show that the fractional binding energy (i.e. $1-E$) of a stable circular orbit in the limit $R \to 6M$ is

$$(1 - 2\sqrt{2}/3) \simeq 0.0572.$$

8. A massive test particle flies past a spherically symmetric star of mass M and Schwarzschild radius $r_s = 2M$ (using units in which $G = c = 1$), which causes a small deflection in the particle's trajectory. Using the radial equation with effective potential $V_{\text{eff}}(r)$ and setting $u = 1/r$, show that

$$\Big(\frac{\mathrm{d}u}{\mathrm{d}\phi}\Big)^2 \;=\; \frac{E^2 - 1}{h^2} \;-\; u^2 \;+\; r_s u\Big(\frac{1}{h^2} + u^2\Big)\,.$$

Now consider a solution of the form $u(\phi) = \sum_{n=0}^{\infty} \epsilon^n u_n(\phi)$, where $\epsilon \ll 1$.

(i) If $r_s u \ll 1$, the form of the trajectory to leading order can be found by setting $r_s = 2M = 0$. Show that the resulting solution is $u_0(\phi) = (1/b)\sin\phi$, with a suitable choice of the coordinate ϕ, where b is a constant. Show also that if the speed of the particle is v when $r = b$, then $h = bv/\sqrt{1 - v^2}$.

(ii) Setting $\epsilon = r_s/b \ll 1$ and taking $u_0(\phi)$ as in (i), find a second order differential equation for $u_1(\phi)$. Hence show that, to order ϵ, the angular deflection is given by

$$\Delta\phi \;\approx\; 2\epsilon\Big(1 + \frac{b^2}{2h^2}\Big) \;=\; \frac{r_s(1 + v^2)}{bv^2}\,.$$

Comment on the limit $v \to 1$.

9. In an attempt to unify Special Relativity and Newtonian Gravity, the orbits in a central potential are calculated using the Euler-Lagrange equations derived from the Lagrangian

$$\mathcal{L} \;=\; -c^2\gamma^{-1} + \frac{GM}{r}$$

where $\gamma = (1 - \mathbf{v}\cdot\mathbf{v}/c^2)^{-1/2}$ (in the obvious notation).

Obtain the orbital equation for motion in the equatorial plane in the form

$$u'' + u \;=\; \gamma\,\ell^{-1}$$

where $u = 1/r$, prime denotes differentiation with respect to ϕ, the usual angular coordinate in the plane, $\ell = h^2/GM$, and h is a conserved quantity, to be specified.

Show that $\gamma = \big(1 + h^2(u'^2 + u^2)/c^2\big)^{1/2}$. Using the approximation $\gamma \approx 1 + \frac{1}{2}h^2(u'^2 + u^2)/c^2$, show that the rate of advance of the perihelion is one sixth of that obtained using the Schwarzschild metric.

10. A classical test of general relativity is the time delay caused to radar signals bounced off planets or satellites. Ignoring curvature a light ray is given by $r \sin\phi = b$. By differentiating this, show that $r^2 d\phi^2 = b^2 dr^2/(r^2 - b^2)$.

Assuming the above relation between $d\phi$ and dr obtain the approximate equation for light rays in Schwarzschild

$$dt = \pm \frac{r}{\sqrt{r^2 - b^2}} \left(1 + \frac{2M}{r} - \frac{Mb^2}{r^3} \right) dr \ ,$$

where $\mathrm{O}\big((M/r)^2\big)$ terms have been neglected. Show that, to this approximation, the time taken to move from $r = b$ to $r = r_1$ is given by

$$\Delta t = \sqrt{r_1^2 - b^2} \ + \ 2M \cosh^{-1}(r_1/b) \ - \ \frac{M}{r_1} \sqrt{r_1^2 - b^2} \, ,$$

and identify the first term on the right hand side.

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