Mathematical Tripos Part II **Michaelmas 2023**
D23 Fluid Dynamics II **Grae Worster**
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Example Sheet 1

Every answer should include at least one relevant sketch

1. Consider the steady two-dimensional inviscid flow

$$\mathbf{u} = (\alpha x - \tfrac{1}{2}\omega y, -\alpha y + \tfrac{1}{2}\omega x).$$

Confirm that this flow is incompressible and find its streamfunction. Show that the streamlines are elliptic or hyperbolic according to whether $|\alpha| \lessgtr \frac{1}{2}|\omega|$.

Evaluate the inertia $\rho\mathbf{u}\cdot\nabla\mathbf{u}$ and find a pressure field to balance it. Discuss the minimal or maximal nature of the pressure at the origin in terms of the streamline pattern.

2. Show that a steady shear flow $\mathbf{u} = [\gamma y, 0, 0]$, where γ is constant, is the sum of a planar extensional flow (whose principal axes should be determined) and a solid-body rotation. Show that the Navier-Stokes equations are satisfied if the pressure is uniform and the body force vanishes. If this shear is maintained in a fluid of dynamic viscosity μ flowing between two plates at $y = 0$ and $y = h$, write down the full stress tensor and use it explicitly to find the forces exerted by the fluid on each of the plates. Calculate the rate of work per unit area needed to be exerted on the top plate to maintain the flow and show that it is equal to the rate of dissipation per unit area internal to the flow.
[*Pay close attention to the direction of the normal vectors on each plate.*]

3. Show that, for the flow $\mathbf{u}$ of an incompressible, viscous fluid in a region V enclosed by a stationary rigid boundary,

$$\int_V \frac{\partial u_i}{\partial x_j}\frac{\partial u_j}{\partial x_i}\,dV = 0.$$

Hence show that the total rate of dissipation of energy $D \equiv 2\mu\int_V e_{ij}e_{ij}\,dV$ can be written as

$$D = \mu\int_V \omega^2\,dV, \qquad \text{where} \quad \boldsymbol{\omega} = \nabla\times\mathbf{u}.$$

[It follows that if the flow is irrotational then there is no dissipation. Why?]

4. A layer of incompressible fluid of density ρ and dynamic viscosity μ flows steadily down a plane inclined at an angle θ to the horizontal, forming a uniform layer of thickness h parallel to the plane. A second layer of fluid, of uniform thickness αh, viscosity $\beta\mu$ and density ρ flows steadily on top of the first layer. Using Cartesian coordinates perpendicular and parallel to the plane, write down the equations of motion in each layer, the boundary conditions on the plane and on the top free surface, and the boundary conditions at the interface between the two layers.

Find the pressure, shear-stress and velocity fields in each layer. Why does the velocity profile in the bottom layer depend on α but not β?

Show that the volume flux (per unit cross-slope width) is

$$\frac{\rho g h^3\sin\theta}{3\mu}\left(1+\frac{3\alpha}{2}\right) \qquad \text{and} \qquad \frac{\rho g(\alpha h)^3\sin\theta}{3\beta\mu}\left(1+\frac{3\beta}{2}\frac{1+2\alpha}{\alpha^2}\right)$$

in the lower and upper layers respectively. Discuss the limits (a) $\alpha \ll 1$, (b) $\beta \ll 1$.
[*Recall from Fluids I that the flux for a single layer of thickness h is $\rho g h^3\sin\theta/3\mu$.*]

5. An incompressible fluid of dynamic viscosity μ flows steadily through a cylindrical tube parallel to the z-axis with velocity $\mathbf{u} = [0, 0, w(x, y)]$, under a uniform pressure gradient $G = -dp/dz$. Show that the Navier-Stokes equations with no body force are satisfied provided

$$\nabla^2 w = -G/\mu,$$

and state the appropriate boundary conditions.

Find w for a tube with an elliptical cross-section with semi-axes a and b. [Hint: consider the function $f(x, y) = \left(1 - x^2/a^2 - y^2/b^2\right)$ and recall the uniqueness of the solution to Poisson's equation with Dirichlet boundary conditions.] Show that the volume flux (i.e. the volume of fluid passing through any section of the tube per unit time) is given by

$$Q = \frac{\pi a^3 b^3 G}{4(a^2 + b^2)\mu}.$$

Now consider the circular case with $a = b$ (so-called Poiseuille flow). Show that the viscous stress on the boundary, $\sigma_{rz} = \mu \partial w/\partial r$, exerts a force that exactly balances the pressure difference exerted across the ends. Further, calculate the dissipation within the tube and show that it is equal to the rate of working against the pressure difference across the ends.

6. Viscous fluid flows with steady velocity $\mathbf{u} = [0, v(r), 0]$ between two infinitely-long, coaxial cylinders $r = a$ and b ($> a$). The inner cylinder rotates with steady angular velocity Ω about its axis, while the outer cylinder is at rest. The pressure varies only in the radial direction. Using the Navier-Stokes equations in cylindrical polar coordinates (see e.g. Appendix 2 of Batchelor or Wikipedia), show that

$$v(r) = Ar + B/r,$$

where the constants A and B are to be determined. Calculate the torque per unit length that must be applied to the inner cylinder to maintain the motion; check the dimensions and the sign of your result. [In polar coordinates (r, ϕ), the component $e_{r\phi}$ of the strain-rate tensor is given by $2e_{r\phi} = r\partial(v/r)/\partial r$ for this flow.]

7. The plane rigid boundary of a semi-infinite domain of a viscous fluid oscillates in its own plane with velocity $U_0 \cos \omega t$. The fluid is at rest at infinity. Find the velocity field. [Hint: use complex notation by writing $\cos \omega t$ as the real part of $e^{i\omega t}$.] Show that the time-averaged rate of dissipation of energy in the fluid is

$$\tfrac{1}{2}\rho U_0^2 \left(\tfrac{1}{2}\nu\omega\right)^{1/2}$$

per unit area of the boundary. Verify that this is equal to the time average of the rate of work of the boundary on the fluid (per unit area).

8. A viscous fluid of kinematic viscosity ν and density ρ is confined between a fixed plate at $y = h$ and a plate at $y = 0$ whose velocity is $[U_0 \cos \omega t, 0, 0]$, where U_0 is a constant. There is no body force and the pressure is independent of x. Explain the physical significance of the dimensionless number $S = \omega h^2/\nu$.

Assuming that the flow remains time-periodic and unidirectional, find expressions for the flow profile and the time-average rate of working Φ per unit area by the plates on the fluid. [Hint: use complex notation and the functions sinh and cosh].

Sketch the velocity profile and evaluate Φ in the limits $S \ll 1$ and $S \to \infty$, and explain why in these limits Φ becomes independent of ω and h respectively.

9. Suppose that the tube in question 5 has as its cross-section the sector of a circle $r < a$, $|\theta| < \beta$ in plane polar coordinates (r, θ). Show that the momentum equation has solution

$$w(r, \theta) = \frac{Gr^2}{4\mu}\left(\frac{\cos 2\theta}{\cos 2\beta} - 1\right) + \sum_{n=0}^{\infty} A_n r^{\lambda_n} \cos \lambda_n \theta,$$

where $\lambda_n = (2n + 1)\pi/2\beta$ and the coefficients A_n are to be found. Determine the asymptotic behaviour of the flow near $r = 0$ [Hint: distinguish the cases $\beta \lessgtr \pi/4$]. Under what circumstances is the flow near $r = 0$ independent of the boundary $r = a$?

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