Mathematical Tripos Part II **Michaelmas 2016**
D23 Fluid Dynamics II **Grae Worster**

Example Sheet 3

Every answer should include at least one relevant sketch

1. A rigid sphere of radius a falls under gravity through a Newtonian fluid of viscosity μ towards a horizontal rigid plane. Use lubrication theory to show that, when the minimum gap h_0 is very small, the speed of approach of the sphere is

$$h_0 W/6\pi\mu a^2,$$

where W is the weight of the sphere corrected for buoyancy.

2. A Newtonian fluid of viscosity μ is forced by a pressure difference Δp through the narrow gap between two parallel circular cylinders of radius a with axes $2a+b$ apart. Show that, provided $b \ll a$ and $\rho b^3 \Delta p \ll \mu^2 a$, the volume flux through the gap per unit length along the axis of the cylinders is approximately

$$\frac{2b^{5/2}\Delta p}{9\pi a^{1/2}\mu},$$

when the cylinders are fixed.

Show that when the two cylinders rotate with angular velocities Ω_1 and Ω_2 in opposite directions (i.e. one rotates $\Omega_1 \mathbf{e}_z$ while the other one $-\Omega_2 \mathbf{e}_z$ where $\mathbf{e}_z$ is the unit vector along the axis of the cylinder), the change in the volume flux is given by

$$\frac{2}{3}ab(\Omega_1+\Omega_2).$$

3. A viscous fluid coats the outer surface of a cylinder of radius a which rotates with angular velocity Ω about its axis, which is horizontal. The angle θ is measured from the horizontal on the rising side. Show that the volume flux per unit length $Q(\theta,t)$ is related to the thickness $h(\theta,t) \ll a$ of the fluid layer by

$$Q = \Omega a h - \frac{g}{3\nu}h^3\cos\theta,$$

and deduce an evolution equation for $h(\theta,t)$.

Consider now the possibility of a steady state with $Q = \text{const}$, $h = h(\theta)$. Show that a steady solution with $h(\theta)$ continuous and 2π-periodic exists only if

$$\Omega a > (9Q^2 g/4\nu)^{1/3}.$$

[Hint: Consider a graph of $\cos\theta$ as a function of h.]

4. A drop of viscous fluid of thickness $h(r,t)$ spreads axisymmetrically on a horizontal surface. Explaining your reasoning carefully with the aid of a diagram, use mass conservation to show that

$$\frac{\partial h}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(rq) = 0,$$

where $q(r,t) = \displaystyle\int_0^h u(r,z,t)\,dz$ in plane polar coordinates (r,θ,z) and u is the radial velocity.

Solve the lubrication equations to determine the partial differential equation governing the evolution of h once the drop has become a thin layer. Look for a similarity solution to the equation and apply a condition that the volume of the drop is constant to determine the radius of the drop $r_N(t)$.

5. A Newtonian fluid with dynamic viscosity μ flows in a shallow container with a free surface at $z=0$. Using cartesian coordinates (x,y,z), the fluid velocity is denoted $(u_x,u_y,u_z) \equiv (\mathbf{u}_H,u_z)$. The base of the container is rigid, and is located at $z = -h(x, y)$. An external horizontal stress $S(x, y)$ is applied at the free surface. Gravity may be neglected. Using lubrication theory, show that the two-dimensional horizontal volume flux $\mathbf{q}(x, y) \equiv \int_{-h}^{0} \mathbf{u}_H dz$ satisfies the equations

$$\nabla \cdot \mathbf{q} = 0, \quad \mu\mathbf{q} = -\frac{1}{3}h^3\nabla p + \frac{1}{2}h^2\mathbf{S},$$

where $p(x, y)$ is the pressure. Find also an expression for the surface velocity $\mathbf{u}_0(x, y) \equiv u_H(x, y, 0)$ in terms of $\mathbf{S}$, $\mathbf{q}$ and h.

6. The walls of a straight two-dimensional channel are porous and separated by a distance d. A Newtonian fluid of viscosity μ is driven along the channel by a pressure gradient $G = -\partial p/\partial x$. At the same time, suction is applied to one wall of the channel providing a cross flow with uniform transverse component of velocity $V > 0$, with fluid being supplied at this rate to the other wall. Calculate the steady velocity and vorticity distributions in the fluid. Sketch them (i) when $Vd/\nu \ll 1$ and (ii) when $Vd/\nu \gg 1$.

7. A Newtonian fluid of viscosity μ fills an annulus $a < r < b$ between a long stationary cylinder $r = b$ and a long cylinder $r = a$ rotating at angular velocity Ω. Looking up the components of the Navier-Stokes equation in cylindrical coordinates, find the axisymmetric velocity field, ignoring end effects.
Suppose now that the two cylinders are porous, and a pressure difference is applied so that there is a radial flow $-Va/r$ in the fluid annulus. Find an expression for the new steady flow around the cylinder when $Va/\nu \neq 2$. Comment on the flow structure when $Va/\nu \gg 1$.
Find the torque (per unit length along the cylinder axis) required to maintain the motion, and show that it is independent of b in the limit $Va/\nu \to \infty$. [Check the dimensions and sign of your result.]

8. Starting from the Navier-Stokes equations for incompressible viscous flow with conservative forces, obtain the vorticity equation

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega}\cdot\nabla\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}.$$

Interpret the terms in the equation.
At time $t = 0$ a concentration of vorticity is created along the z-axis, with the same circulation Γ around the axis at each z. The fluid is viscous and incompressible, and for $t > 0$ has only an azimuthal velocity denoted v. Show that there is a similarity solution of the form $vr/\Gamma = f(\eta)$, where $r = (x^2 + y^2)^{1/2}$ and η is a suitable similarity variable. Furthermore, show that all conditions are satisfied by

$$f(\eta) = \tfrac{1}{2\pi}(1 - e^{-\eta^2}), \qquad \eta = r/2\sqrt{\nu t}.$$

Show also that the total vorticity in the flow remains constant at Γ for all $t > 0$. Sketch v as a function of r.

9. Calculate the vorticity $\boldsymbol{\omega}$ associated with the velocity field

$$\mathbf{u} = (-\alpha x - yf(r, t),\ -\alpha y + xf(r, t),\ 2\alpha z)\,,$$

where α is a positive constant, and $f(r, t)$ depends on $r = (x^2+y^2)^{1/2}$ and time t. Show that the velocity field represents a dynamically possible motion if $f(r, t)$ satisfies

$$2f + r\frac{\partial f}{\partial r} = A\gamma(t)e^{-\gamma(t)r^2},$$

where

$$\gamma(t) = \frac{\alpha}{2\nu}\left(1 \pm e^{-2\alpha(t-t_0)}\right)^{-1},$$

and A and t_0 are constants.
Show that, in the case where the minus sign is taken, γ is approximately $1/[4\nu(t - t_0)]$ when t only just exceeds t_0. Which terms in the vorticity equation dominate when this approximation holds?

Please send comments to mgw1@cam.ac.uk.