Example Sheet 1: Sound Waves

1. *Plane waves and radiation.* A thin piston executes very small oscillations about $x = 0$ in a long straight fluid-filled tube with cross-sectional area A and rigid walls aligned with the x-axis. Given the piston velocity $\dot{X}(t)$, find the velocity potential $\phi(x,t)$ for the (linearised) sound waves generated in $x > 0$ and $x < 0$ (linearising $X \approx 0$). Show that if $x_0 > 0$ the total power AI_x radiated across $x = x_0$ is

$$\big(q(t - x_0/c_0)\big)^2 c_0/\rho_0 A, \tag{1}$$

where $q(t) = \rho_0 A\dot{X}(t)$ is the rate at which mass is displaced on one side of the piston. What is the corresponding result for $x_0 < 0$? [*Later in the course, we will analyse the effects of nonlinearity.*]

2. *Reflection and transmission.* An interface at $x = 0$ separates fluid of density ρ_0 and sound speed c_0 in $x < 0$ from fluid of density ρ_1 and sound speed c_1 in $x > 0$. A plane harmonic sound wave is incident from $x < 0$ with wavevector $\mathbf{k} = (k, 0, 0)$ and amplitude A (of its pressure perturbation). What is the frequency ω and the wavevector $\mathbf{k}'$ of the transmitted sound wave in $x > 0$?

Write down the form of the pressure perturbation in $x < 0$ and $x > 0$, find the corresponding velocity potential and state the interfacial boundary conditions. Hence find the amplitudes of the reflected and transmitted waves.

Assume wlog that $A = 1$. Verify that the time-averaged acoustic energy flux is conserved. When is all the energy flux transmitted? How much is reflected if $\rho_0 \gg \rho_1$ and $c_0 \approx c_1$?

3. *Evanescent waves near an interface.* Find solutions to the wave equation of the form

$$\phi(x, y, t) = \exp(ikx - i\omega t)f(y)\ , \tag{2}$$

for the case $k > \omega/c_0 > 0$. Hence find the solution in $y \geqslant 0$ in which there is no disturbance as $y \to \infty$ and waves are forced by the inhomogenous boundary condition

$$v = \mathrm{Re}\,[v_0 \exp(ikx - i\omega t)] \quad \text{on} \quad y = 0\ .$$

Here $\nabla\phi = (u, v, 0)$ and v_0 is a real constant. Over what lengthscale do the waves decay away from the boundary?

Calculate the time-averaged acoustic energy flux $\langle\, \mathbf{I}\,\rangle$ and verify that:
(a) the energy flux perpendicular to the boundary $y = 0$ satisfies $\langle\, I_y\,\rangle = 0$;
(b) the energy flux parallel to the boundary satisfies $\langle\, I_x\,\rangle = c\,\langle\, E\,\rangle$ at any position y, where E is the acoustic energy density and $c = \omega/k$ is the phase velocity in the x-direction. [*Since $c < c_0$, the disturbance and its energy travel subsonically along the boundary.*]

*Assuming that surface tension and gravity are negligible, determine whether a non-zero solution can exist in which evanescent sound waves propagate along both sides of an (unforced) interface between two fluids with different physical properties in $y < 0$ and $y > 0$,

4. *Acoustic waveguide.* Find solutions to the wave equation of the form (2) for a region $0 < y < h$ with a rigid boundary at $y = 0$ and a free boundary at $y = h$. (Take $\omega > 0$, but make no *a priori* assumption about k.) Show that a wave can propagate in the x-direction only if ω exceeds a critical value ω_c. What happens if a disturbance is generated at $x = 0$ with frequency $\omega < \omega_c$?

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5. *Spherical waves and radiation.* Explain why the general spherically symmetric solution $\phi(r,t)$ to the wave equation can be written as

$$\phi = \frac{-1}{4\pi\rho_0}\left(\frac{q(t-r/c_0)}{r} + \frac{Q(t+r/c_0)}{r}\right) , \tag{3}$$

where q and Q are arbitrary functions. Assume from now on that there are only outgoing waves. Calculate the radial velocity u_r and the pressure perturbation $\tilde{p}$.

(a) By considering the volume flux through a sphere of radius ϵ as $\epsilon \to 0$, show that $q(t)$ is the mass flux out of $r = 0$. Show also that ϕ actually satisfies

$$\nabla^2\phi - c_0^{-2}\partial^2\phi/\partial t^2 = q(t)\delta(\mathbf{x})/\rho_0 , \tag{4}$$

where δ is the Dirac delta function. (*Hint*: integrate (4) over $r \leqslant \epsilon$ and let $\epsilon \to 0$.) [*The notation in (3) is standard and motivated by the meaning of q. In some of the detailed calculations below, you may prefer to write $q(t)/4\pi\rho_0 = f(t)$ for brevity.*]

(b) Show that in the far-field, i.e. for 'large' r, the kinetic energy density K, the potential energy density W, and the acoustic-energy flux $\mathbf{I} = \tilde{p}\mathbf{u}$, approximately satisfy the same equations, $K = W$ and $I = (K + W)c_0$, as in a plane wave. Similarly, show that the total power radiated across a 'large' sphere of radius R is approximately

$$\big(\dot{q}(t - R/c_0)\big)^2/4\pi\rho_0 c_0 . \tag{5}$$

[*The solution (3) with $Q = 0$ is called a point source, or an acoustic monopole of strength $\dot{q}(t)$.*]

What does 'large r' mean for a time-harmonic source with $q(t) = \mathrm{Re}(q_0 e^{i\omega t})$?

(c) For the same time-harmonic source, show that $\langle K \rangle / \langle W \rangle \sim (c_0/r\omega)^2$ as $r \to 0$ and find both I_r and $\langle I_r \rangle$ in the same limit. Comment on these results. Compare (1) with (5) for the case $A \ll (c_0/\omega)^2$. (What does this condition mean physically?) [*This is one of the principles behind the 'horn loudspeaker'.*]

6. *Harmonic series.* Explain why (3) describes a possible acoustic disturbance in a conical tube of any cross-sectional shape. Model an oboe (with all the finger-holes closed) as a small-angle conical tube of length ℓ: at the narrow end the cross-sectional area is effectively zero and $\tilde{p}$ is finite; the larger end is open and $\tilde{p}$ may be assumed to be zero. [*This is a good approximation only if the radius of the larger end is much less than c_0/ω.*] Show that the instrument has a set of normal modes (i.e. standing-wave solutions of the form $R(r)e^{-i\omega t}$) with frequencies

$$\omega\ell/c_0 = n\pi \quad (n \in \mathbb{Z}) . \tag{†}$$

If, instead, the larger end is closed, so that the radial velocity is zero there, show that the corresponding normal-mode frequencies are the solutions of

$$\omega\ell/c_0 = \tan(\omega\ell/c_0) . \tag{‡}$$

Find approximate solutions of (‡) in the high-frequency limit. [*The set of frequencies* (†) *forms a musical 'harmonic series', while the set* (‡) *does not.*]

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7. *An oscillating bubble* (Tripos 93124). A bubble makes small spherically symmetric oscillations in a compressible inviscid fluid. When the radius $a(t)$ is perturbed slightly from its mean value a_0, the internal dynamics of the bubble are such that the bubble exerts a perturbation pressure $-\beta(a-a_0)$ on the fluid, where β is a constant. Derive the linearised equation of motion for the oscillations

$$\rho_0 a_0 \ddot{a} + \frac{\beta a_0}{c_0}\dot{a} + \beta(a-a_0) = 0 \ ,$$

where ρ_0 is the undisturbed density of the fluid and c_0 is the sound speed (you may quote results from question 5). What is the mechanism of energy loss from the oscillations represented by the 'damping' term in this ODE for a?

8*. *Images.* Explain briefly how the method of images can be used to find the sound field produced by a point source placed either near a plane rigid boundary, or in the corner between two plane rigid boundaries at right-angles (i.e. find an image system that satisfies the boundary conditions).

For each case, use the results of 5(b) to write down an approximation to the time-averaged total power radiated by a time-harmonic point source if the distance of the source to the boundaries is much less than a wavelength. Will a whistle sound louder if blown near a wall?

9*. *Source with boundaries.* A point source (monopole) is placed at $\mathbf{x} = \mathbf{x}_0$ inside a straight semi-infinite tube aligned along the positive x-axis with a closed end at $x = 0$ and cross-sectional area A. By integrating (4) with $\delta(\mathbf{x})$ replaced by $\delta(\mathbf{x} - \mathbf{x}_0)$, and using the boundary conditions, show that the cross-sectional average potential

$$\overline{\phi}(x,t) = \frac{1}{A}\int\!\!\int \phi\, dy\, dz$$

satisfies

$$\frac{\partial^2 \overline{\phi}}{\partial x^2} - \frac{1}{c_0^2}\frac{\partial^2 \overline{\phi}}{\partial t^2} = \frac{q(t)}{\rho_0 A}\delta(x - x_0).$$

[*Hint*: Recall $\delta(\mathbf{x}) = \delta(x)\delta(y)\delta(z)$.] What conditions should be imposed on $\overline{\phi}$ and $\partial\overline{\phi}/\partial x$ at $x = 0$, $x = x_0$ and as $x \to \infty$?

For a time-harmonic source, $q(t) = q_0 e^{i\omega t}$ (real part understood), show that

$$\overline{\phi} = \frac{i c_0 q_0}{\omega \rho_0 A}\frac{\exp\{i\omega[t - (x - x_0)/c_0]\}}{1 + i\tan(\omega x_0/c_0)} \quad \text{in } x > x_0.$$

If $A \ll (c_0/\omega)^2$, why it is reasonable to assume that the sound field is almost one-dimensional (i.e. $\phi(\mathbf{x}, t) \approx \overline{\phi}(x, t)$) except near $x = x_0$? Making this assumption, show that if $x_0 \ll c_0/\omega$ (what does this mean physically?) then the time-averaged power radiated across a section in $x > x_0$ is the same as the time-average of (1) for this $q(t)$ and a large factor $4\pi c_0^2/\omega^2 A$ bigger than (5).

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