Dynamical Systems

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Dynamical Systems: Course Summary

Informal Introduction

Need for geometric and analytic arguments. Fixed points, periodic orbits and stability. Structural instability and bifurcations. Chaos.

Chapter 1: Basic Definitions

1.1 Notation

Phase space, order of a system, autonomous ODEs, maps.

1.2 Initial value problems

Examples of non-uniqueness and finite-time blow-up.

1.3 Trajectories and flows

1.4 Trajectories, orbits, invariant & limit sets

Orbit/trajectory, forwards orbit, invariant set, fixed pt, periodic pt, limit cycle, homoclinic & heteroclinic orbits, ω- & α-limit sets, N-cycles.

1.5 Topological equivalence and structural stability of flows

Chapter 2: Fixed Points in $\mathbb{R}^2$

2.1 Linearization

The Jacobian A.

2.2 Classification of fixed points

Canonical forms. Saddle point, stable/unstable nodes, stable/unstable foci, stellar node, improper node, centre, line of fixed pts. Hamiltonian systems. Effect of linear perturbations. Hyperbolic and nonhyperbolic fixed points. Sources, sinks & saddles.

2.3 Effect of nonlinear terms

2.3.1 Stable, unstable and centre subspaces. Stable and unstable manifolds. Stable manifold theorem. 2.3.2 Nonhyperbolic cases.

2.4 Sketching phase planes/portraits

General procedure. Worked example.

Chapter 3: Stability

3.1 Definitions

Lyapunov, quasi-asymptotic and asymptotic stability of fixed pts and invariant sets.

3.2 Lyapunov functions

Lyapunov function. Lyapunov's 1st theorem, La Salle's invariance principle. Domain of stability, global stability. General method for Lyapunov functions.

3.3 Bounding functions

Chapter 4: Existence and Stability of Periodic Orbits in $\mathbb{R}^2$

4.1 Poincaré index test

4.2 Dulac's Criterion (& the divergence test)

4.3 Poincaré–Bendixson Theorem

***Proof** and examples*

4.4 Near-Hamiltonian flows

Energy balance (Melnikov) method

4.5 Stability of periodic orbits

Floquet theory. Hyperbolic and nonhyperbolic orbits. Use of $\nabla \cdot \mathbf{f}$ to find stability in $\mathbb{R}^2$.

4.6 Example: van der Pol oscillator

Chapter 5: Bifurcations of Flows

5.1 Introduction

Bifurcation, bifurcation point, bifurcation diagram.

5.2 Centre manifold and extended centre manifold

Centre manifold theorem. Evolution on the centre manifold. Extended system.

5.3 Stationary bifurcations ($\lambda = 0$)

Normal forms: saddle-node, transcritical, subcritical pitchfork & supercritical pitchfork bifurcations. Classification and reduction to normal form. Structural instability of the transcritical and pitchfork cases.

5.4 Oscillatory/Hopf bifurcation ($\lambda = \pm i\omega$)

Normal form for subcritical and supercritical Hopf bifurcations.

5.5 **Bifurcations of periodic orbits**

Chapter 6: Fixed Points and Bifurcations of Maps

6.1 Introduction

Logistic, tent, rotation and sawtooth (shift) maps

6.2 Fixed points, cycles and stability

Definitions. Stability. Nonhyperbolicity for maps.

6.3 Local bifurcations in 1D maps

6.3.1 $\lambda = 1$*: saddle-node, transcritical, subcritical pitchfork* & *supercritical pitchfork bifurcations. 6.3.2* $\lambda = -1$*: period-doubling bifurcation.*

6.4 Example: The logistic map

Chapter 7: Chaos

7.1 Introduction

Definitions of SDIC and TT. Chaos (Devaney). Horseshoe. Chaos (Glendinning).

7.2 The sawtooth map (Bernoulli shift)

7.3 Horseshoes, symbolic dynamics and the shift map

7.4 Period-3 implies chaos

7.5 The existence of N-cycles

Period-3 implies all periods. Sharkovsky's theorem.

7.6 The tent map

Attracting set. Rescaling of subtents for F^2*. Chaos for* $1 < \mu \leq 2$.

7.7 The logistic map

*Period-doubling cascades and chaos. **7.7.1 Unimodal maps. Periods* 2^{2^n} *only or chaos. 7.7.2 Rescaling and Feigenbaum's constant.***

Introduction

Books

There are many excellent texts.

- P.A.Glendinning *Stability, Instability and Chaos* [CUP].
  A very good text written in clear language.
- D.K.Arrowsmith & C.M.Place *Introduction to Dynamical Systems* [CUP].
  Also very good and clear, covers a lot of ground.
- R.Grimshaw *An Introduction to Nonlinear Ordinary Differential Equations* [CRC Press].
  Very good on stability of periodic solutions. Quite technical in parts.
- P.G.Drazin *Nonlinear Systems* [CUP].
  Covers a great deal of ground in some detail. Good on the maps part of the course. Could be the book to go to when others fail to satisfy.
- D.W.Jordan & P.Smith *Nonlinear Ordinary Differential Equations* [OUP].
  A bit long in the tooth and not very rigorous but has some very useful material especially on perturbation theory.
- S.H.Strogatz *Nonlinear Dynamics and Chaos* [Perseus Books, Cambridge, MA.]
  An excellent informal treatment, emphasising applications. Inspirational!

Motivation

Why do we study nonlinear phenomena? Because almost all physical phenomena are describable by of nonlinear dynamical systems (either differential equations or maps). Of course some things like simple harmonic motion can be understood on linear theory: but to know how the period of a pendulum changes with amplitude we must solve a nonlinear equation. Solving a linear system usually requires a simple set of tasks, like determining eigenvalues. Nonlinear systems have an amazingly rich structure, and most importantly they do not in general have analytical solutions, or at least none expressible in terms of elementary functions. Thus in general we rely on a *geometric* approach, which allows the determination of important characteristics of the solution without the need for explicit solution. Much of the course will be taken up with such ideas.

We also study the *stability* of various simple solutions. A solution (steady or periodic state) is not of much use if small perturbations destroy it (an example: a pencil balanced exactly on its point). So we need to know what happens to the solution near such a special solution. This involves linearizing, which allows the classification of fixed and periodic points(corresponding to oscillations). We shall also develop *perturbation methods*, which allow us to find good approximations to solutions of systems that are near simple solutions that we understand.

Many nonlinear systems depend on one or more *parameters*. Examples include the simple equation $\dot{x} = \mu x - x^3$, where the parameter μ can take positive or negative values. If $\mu > 0$ there are three stationary points, while if $\mu < 0$ there is only one. The point $\mu = 0$ is called a *bifurcation point*, and we shall see that we can classify bifurcations and develop a general method for determining the solutions near such points.

Most of the systems we shall look at are of second order, and we shall see that these have relatively simple long-time solutions (fixed and periodic points, essentially). In the last part of the course we shall look at some aspects of third order (and time-dependent second order) systems, which can exhibit "chaos". These systems are usually treated by the study of maps (of the line or the plane) which can be identified with the dynamics if the differential system. Maps can be treated in a rigorous manner and there are some remarkable theorems (such as Sharkovsky's on the order of appearance of periodic orbits in one-humped maps of the interval) that can be proved.

A simple example of a map is the *Lotka-Volterra system* describing two competing populations (e.g. r=rabbits, s=sheep):

$$\dot{r} = r(a - br - cs), \quad \dot{s} = s(d - er - fs)$$

where a, b, c, d, e, f are (positive in this example) constants. This is a *second order system* which is *autonomous* (time does not appear explicitly). The system lives in a *state space* or *phase space* $(r, s) \in [0, \infty) \times [0, \infty)$. We regard r, s as continuous functions of time: some people call this a *flow* (as opposed to a map where numbers vary discontinuously - which might be better for this example!)

Typical analysis looks at *fixed points*. These are at $(r, s) = (0, 0)$, $(r, s) = (0, d/f)$, $(r, s) = (a/b, 0)$ and a solution with $r, s \neq 0$ as long as $bf \neq ce$. Assuming, as can be proved, that at long times the solution tends to one of these, we can look at local approximations near the fixed points. Near $(0, d/f)$, write $u = s - d/f$, then approximately $\dot{r} = r(a - cd/f)$, $\dot{u} = -du - der/f$, so the solution tends to this point ($r = u = 0$) if $s < d/f$ (so this fixed point is *stable*), but not otherwise. The concept of stability if more involved than naive ideas would suggest and so we will be investigating the nature of stability. In fact there are three different *phase portraits* depending on the parameters. The solutions follow lines in the phase space called *trajectories*. We use *bifurcation theory* to study the change in stability as parameters are varied.

Other Lotka-Volterra models have different properties, for example the struggle between sheep s and wolves w:

$$\dot{w} = w(-a + bs), \quad \dot{s} = s(c - dw)$$

This system turns out to have *periodic orbits*.

In two dimensions periodic orbits are common for topological reasons, so we also look at their stability. Consider the system

$$\dot{x} = -y + x(\mu - x^2 - y^2), \quad \dot{y} = x + y(\mu - x^2 - y^2)$$

In polar coordinates r, θ, $x = r\cos\theta$, $y = r\sin\theta$ we have

$$\dot{r} = \mu r - r^3, \quad \dot{\theta} = 1$$

The case $\mu = 0$ is special since there are infinitely many periodic orbits. This is *non-hyperbolic* or not *structurally stable*. Any non-zero value of μ makes a change to the nature of the system.

The stability of periodic orbits can be cast in terms of maps. If an orbit (e.g. in a three-dimensional phase space) crosses a plane at a point $\boldsymbol{x}_n$ and then crosses again at $\boldsymbol{x}_{n+1}$ this defines a map of the plane into itself (the *Poincaré map*).

Maps also arise naturally as approximations to flows,e.g. the equation $\dot{x} = \mu x - x^3$ can be approximated using Euler's method (with $x_n = x(n\,dt)$) to give $x_{n+1} = x_n(\mu dt + 1) - x_n^3 dt$.

Poincaré maps arising from flows in 2D are dull, but for 3D flows they become maps of the plane and can have many interesting properties including *chaos*. Even one dimensional maps can have chaotic behaviour in general. Examples include the *Lorenz equations* ($\dot{x} = \sigma(y - x)$, $\dot{y} = rx - y - xz$, $\dot{z} = -bz + xy$, and the *logistic map* $x_{n+1} = \mu x_n(1 - x_n)$.

1 Nonlinear Differential Equations

1.1 Elementary concepts

We need some notation to describe our equations.

Define a **State Space** (or **Phase Space**) $E \subseteq \mathbb{R}^n$ (E is sometimes denoted by X). Then the **state** of the system is denoted by $\boldsymbol{x} \in E$. The state depends on the **time** t and the (ordinary) differential equation gives a rule for the evolution of $\boldsymbol{x}$ with t:

$$\dot{\boldsymbol{x}} \equiv \frac{d\boldsymbol{x}}{dt} = \boldsymbol{f}(\boldsymbol{x}, t)\,, \tag{1}$$

where $\boldsymbol{f} : E \times \mathbb{R} \to E$ is a vector field.

If $\dfrac{\partial \boldsymbol{f}}{\partial t} \equiv 0$ the equation is **autonomous**. The equation is of **order $\boldsymbol{n}$**. N.B. a system of n first order equations as above is equivalent to an n^{th} order equation in a single dependent variable. If $d^n x/dt^n = g(x, dx/dt, \ldots, d^{n-1}x/dt^{n-1})$ then we write $\boldsymbol{y} = (x, dx/dt, \ldots, d^{n-1}x/dt^{n-1})$ and $\dot{\boldsymbol{y}} = (y_2, y_3, \ldots, y_n, g)$.

Non-autonomous equations can be made (formally) autonomous by defining $\boldsymbol{y} \in E \times \mathbb{R}$ by $\boldsymbol{y} = (\boldsymbol{x}, t)$, so that $\dot{\boldsymbol{y}} = \boldsymbol{g}(\boldsymbol{y}) \equiv (\boldsymbol{f}(\boldsymbol{y}), 1)$.

Example 1 *Second order system* $\ddot{x} + \dot{x} + x = 0$ *can be written* $\dot{x} = y$, $\dot{y} = -x - y$, *so* $(x, y) \in \mathbb{R}^2$.

1.2 Initial Value Problems

Typically, seek **solutions** to (1) understood as an **initial value problem**:

Given an initial condition $\boldsymbol{x}(t_0) = \boldsymbol{x}_0$ ($\boldsymbol{x}_0 \in E$, $t_0 \in I \subseteq \mathbb{R}$), find a differentiable function $\boldsymbol{x}(t)$ for $t \in I$ which remains in E for $t \in I$ and satisfies the initial condition and the differential equation.

For an autonomous system we can alternatively define the solution in terms of a **flow** ϕ_t:

Definition 1 *(Flow) $\phi_t(\boldsymbol{x})$ s.t. $\phi_t(\boldsymbol{x}_0)$ is the solution at time t of $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ starting at $\boldsymbol{x}_0$ when $t = 0$ is called the **flow** through $\boldsymbol{x}_0$ at $t = 0$. Thus $\phi_0(\boldsymbol{x}_0) = \boldsymbol{x}_0$, $\phi_s(\phi_t(\boldsymbol{x}_0)) = \phi_{s+t}(\boldsymbol{x}_0)$ etc. (Continuous semi-group). We sometimes write $\phi_t^{\boldsymbol{f}}(\boldsymbol{x}_0)$ to identify the particular dynamical system leading to this flow.*

Does such a solution exist? And is it unique?

Existence is guaranteed for many sensible functions by the ****Cauchy-Peano theorem****:

Theorem 1 *(Cauchy-Peano). If $\boldsymbol{f}(\boldsymbol{x}, t)$ is continuous and $|\boldsymbol{f}| < M$ in the domain $\mathcal{D} : \{|t - t_0| < \alpha,\ |\boldsymbol{x} - \boldsymbol{x}_0| < \beta\}$, then the initial value problem above has a solution for $|t - t_0| < \min(\alpha, \beta/M)$.*

But *uniqueness* is guaranteed only for stronger conditions on $\boldsymbol{f}$.

Example 2 *Unique solution: $\dot{x} = |x|$, $x(t_0) = x_0$. Then $x(t) = x_0 \mathrm{e}^{t-t_0}$ $(x_0 > 0)$, $x(t) = x_0 \mathrm{e}^{t_0 - t}$ $(x_0 < 0)$, $x(t) = 0$ $(x_0 = 0)$. Here f is not differentiable, but it is continuous.*

Example 3 *Non-unique solution: $\dot{x} = |x|^{\frac{1}{2}}$, $x(t_0) = x_0$. We still have f continuous. Solving gives $x(t) = (t + c)^2/4$ $(x > 0)$ or $x(t) = -(c - t)^2/4$ $(x < 0)$. So for $x_0 > 0$ we have $x(t) = (t - t_0 + \sqrt{4x_0})^2/4$ $(t > t_0)$. However for $x_0 = 0$ we have **two** solutions: $x(t) = 0$ (all t) and $x(t) = (t - t_0)|(t - t_0)|/4$ $(t \geq t_0)$.*

Why are these different? Because in second case derivatives of $|x|^{\frac{1}{2}}$ are not bounded at the origin. To guarantee uniqueness of solutions need stronger property than continuity; function to be **Lipschitz**.

Definition 2 *(Lipschitz property). A function $\boldsymbol{f}$ defined on a subset of $\mathbb{R}^n$ satisfies a **Lipschitz condition** at a point $\boldsymbol{x}_0$ with Lipschitz constant L if $\exists(L, a)$ such that $\forall \boldsymbol{x}, \boldsymbol{y}$ with $|\boldsymbol{x} - \boldsymbol{x}_0| < a, |\boldsymbol{y} - \boldsymbol{x}_0| < a$, $|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})| \leq L|\boldsymbol{x} - \boldsymbol{y}|$.*

Note that **Differentiable** $\rightarrow$ **Lipschitz** $\rightarrow$ **Continuous**.

We can now state the result (discussed in Part IA also):

Theorem 2 *(Uniqueness theorem). Consider an initial value problem to the system (1) with $\boldsymbol{x} = \boldsymbol{x}_0$ at $t = t_0$. If $\boldsymbol{f}$ satisfies a Lipschitz condition at $\boldsymbol{x}_0$ then the solution $\phi_{t-t_0}(\boldsymbol{x}_0)$ exists and is unique and continuous in a neighbourhood of $(\boldsymbol{x}_0, t_0)$*

Note that uniqueness and continuity do not mean that solutions exist for all time!

Example 4 *(Finite time blowup). $\dot{x} = x^3$, $x \in \mathbb{R}$, $x(0) = 1$. This is solved by $x(t) = 1/\sqrt{1 - 2t}$, so $x \to \infty$ when $t \to \frac{1}{2}$.*

This does not contradict earlier result [why not?] .

From now on consider continuous functions $\boldsymbol{f}$ unless stated otherwise.

1.3 Trajectories and Flows

Consider the o.d.e. $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ with $\boldsymbol{x}(0) = \boldsymbol{x}_0$, or equivalently the **flow** $\phi_t(\boldsymbol{x}_0)$

Definition 3 *(Orbit). The **orbit** of ϕ_t through $\boldsymbol{x}_0$ is the set $\mathcal{O}(\boldsymbol{x}_0) \equiv \{\phi_t(\boldsymbol{x}_0) : -\infty < t < \infty\}$. This is also called the **trajectory** through $\boldsymbol{x}_0$.*

Definition 4 *(Forwards orbit). The **forwards orbit** of ϕ_t through $\boldsymbol{x}_0$ is $\mathcal{O}^+(\boldsymbol{x}_0) \equiv \{\phi_t(\boldsymbol{x}_0) : t \geq 0\}$; **backwards orbit** $\mathcal{O}^-$ defined similarly for $t \leq 0$.*

Note that flows and maps can be linked by considering $\boldsymbol{x}_{n+1} = \phi_{\delta t}(\boldsymbol{x}_n)$ In this course we adopt a *geometric* viewpoint: rather than solving equations in terms of "elementary" (a.k.a. tabulated) functions, look for general properties of the solutions. Since almost all equations cannot be solved in terms of elementary functions, this is more productive!

1.4 Invariant and Limit Sets

Work by considering the **phase space** E, and the flow $\phi_t(\boldsymbol{x}_0)$, considered as a trajectory (directed line) in the phase space. we are mostly interested in special sets of trajectories, as long-time limits of solutions from general initial conditions. These are called **invariant sets**.

Definition 5 *(Invariant set). A set of points $\Lambda \in E$ is **invariant under $\boldsymbol{f}$** if $\boldsymbol{x} \in \Lambda \Rightarrow \mathcal{O}(\boldsymbol{x}) \in \Lambda$. (Can also define forward and backward invariant sets in the obvious way).*

Clearly $\mathcal{O}(\boldsymbol{x})$ is invariant. Special cases are;

Definition 6 *(Fixed point). The point* $\boldsymbol{x}_0$ *is a* ***fixed point (equilibrium, stationary point, critical point)*** *if* $\boldsymbol{f}(\boldsymbol{x}_0) = 0$*. Then* $\boldsymbol{x} = \boldsymbol{x}_0$ *for all time and* $\mathcal{O}(\boldsymbol{x}_0) = \boldsymbol{x}_0$*.*

Definition 7 *(Periodic point). A point* $\boldsymbol{x}_0$ *is a* ***periodic point*** *if* $\phi_T(\boldsymbol{x}_0) = \boldsymbol{x}_0$ *for some* $T > 0$*, but* $\phi_t(\boldsymbol{x}_0) \neq \boldsymbol{x}_0$ *for* $0 < t < T$*. the set* $\{\phi_t(\boldsymbol{x}_0) : 0 \leq t < T\}$ *is called a* ***periodic orbit*** *through* $\boldsymbol{x}_0$*.* T *is the* ***period*** *of the orbit. If a periodic orbit* $\mathcal{C}$ *is* ***isolated****, so that there are no other periodic orbits in a sufficiently small neighbourhood of* $\mathcal{C}$*, the periodic orbit is called a* ***limit cycle****.*

Example 5 *(Family of periodic orbits). Consider*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y^3 \\ x^3 \end{pmatrix}$$

This has solutions of form $x^4 + y^4 =$ *const., so all orbits except the fixed point at the origin are periodic.*

Example 6 *(Limit cycle). Now consider*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y + x(1 - x^2 - y^2) \\ x + y(1 - x^2 - y^2) \end{pmatrix}$$

Here we have $\dot{r} = r(1 - r^2)$*, where* $r^2 = x^2 + y^2$*, and there are no fixed points except the origin, so there is a unique limit cycle* $r = 1$*.*

Definition 8 *(Homoclinic and heteroclinic orbits). If $\boldsymbol{x}_0$ is a fixed point and $\exists \boldsymbol{y} \neq \boldsymbol{x}_0$ such that $\phi_t(\boldsymbol{y}) \to \boldsymbol{x}_0$ as $t \to \pm\infty$, then $\mathcal{O}(\boldsymbol{y})$ is called a* ***homoclinic orbit****. If there are two fixed points $\boldsymbol{x}_0, \boldsymbol{x}_1$ and $\exists \boldsymbol{y} \neq \boldsymbol{x}_0, \boldsymbol{x}_1$ such that $\phi_t(\boldsymbol{y}) \to \boldsymbol{x}_0\,(t \to -\infty)$, $\phi_t(\boldsymbol{y}) \to \boldsymbol{x}_1\,(t \to +\infty)$ then $\mathcal{O}(\boldsymbol{y})$ is a* ***heteroclinic orbit****. A closed sequence of heteroclinic orbits is called a* ***heteroclinic cycle*** *(sometimes also called heteroclinic orbit!)*

When the phase space has dimension greater than 2 can get more exotic invariant sets.

Example 7 *(2-Torus). let θ_1, θ_2 be coordinates on the surface of a 2-torus, such that $\dot{\theta}_1 = \omega_1$, $\dot{\theta}_2 = \omega_2$. If ω_1, ω_2 are not rationally related the trajectory covers the whole surface of the torus.*

Example 8 *(Strange Attractor). Anything more complicated than above is called a* ***strange attractor****. Examples include the* ***Lorenz attractor****.*

We have to be careful in defining how invariant sets arise as limits of trajectories. Not enough to have definition like "set of points $\boldsymbol{y}$ s.t. $\phi_t(\boldsymbol{x}) \to \boldsymbol{y}$ as $t \to \infty$", as that does not include e.g. periodic orbits. Instead use the following:

Definition 9 *(Limit set). The* ***$\boldsymbol{\omega}$-limit set of $\boldsymbol{x}$****, denoted by $\omega(\boldsymbol{x})$ is defined by $\omega(\boldsymbol{x}) \equiv \{\boldsymbol{y} : \phi_{t_n}(\boldsymbol{x}) \to \boldsymbol{y}$ for some sequence of times $t_1, t_2, \ldots, t_n, \ldots \to \infty\}$. Can also define* ***$\alpha$-limit set*** *by sequences $\to -\infty$.*

The ω-limit set $\omega(\boldsymbol{x})$ has nice properties when $\mathcal{O}(\boldsymbol{x})$ is bounded: In particular, $\omega(\boldsymbol{x})$ is:

(a) Non-empty [every sequence of points in a closed bounded domain has at least one accumulation point] (b) Invariant under $\boldsymbol{f}$ [Obvious from definition] (c) Closed [think about *not* being in $\omega(\boldsymbol{x})$] and bounded (d) Connected [if disconnected, $\exists$ an sequence of times for which $\boldsymbol{x}(t)$ does not tend to any of the disconnected parts of $\omega(\boldsymbol{x})$]

1.5 Topological equivalence and structural stability

What do we mean by saying that two flows (or maps) have essentially the same (topological/geometric) structure? Or that the structure of a flow changes at a bifurcation?

Definition 10 *(Topological Equivalence). Two flows $\phi_t^{\boldsymbol{f}}(\boldsymbol{x})$ and $\phi_t^{\boldsymbol{g}}(\boldsymbol{y})$ are* ***topologically equivalent*** *if there is a homeomorphism $\boldsymbol{h}(\boldsymbol{x}) : E^{\boldsymbol{f}} \to E^{\boldsymbol{g}}$ (i.e. a continuous bijection with continuous inverse) and time-increasing function $\tau(\boldsymbol{x}, t)$ (i.e. a continous, monotonic function of t) with*

$$\phi_t^{\boldsymbol{f}}(\boldsymbol{x}) = \boldsymbol{h}^{-1} \circ \phi_\tau^{\boldsymbol{g}} \circ \boldsymbol{h}(\boldsymbol{x}) \quad \textit{and} \quad \tau(\boldsymbol{x}, t_1 + t_2) = \tau(\boldsymbol{x}, t_1) + \tau(\phi_{t_1}^{\boldsymbol{f}}, t_2)$$

i.e. it is possible to find a map $\boldsymbol{h}$ from one phase space to the other, and a map τ from time in one phase space to time in the other, in such a way that the evolution of the two systems are the same. Clearly topological equivalence maps fixed points to fixed points, and periodic orbits to periodic orbits – though not necessarily of the same period.

Example 9 *The dynamical systems*

$$\begin{array}{c} \dot{r} = -r \\ \dot{\theta} = 1 \end{array} \quad \textit{and} \quad \begin{array}{c} \dot{\rho} = -2\rho \\ \dot{\psi} = 0 \end{array}$$

are topologically equivalent with $\boldsymbol{h}(\boldsymbol{0}) = \boldsymbol{0}$, $\boldsymbol{h}(r, \theta) = (r^2, \theta + \ln r)$ *for* $r \neq 0$ *in polar coordinates, and* $\tau(\boldsymbol{x}, t) = t$. *To show this, integrate the ODEs to get*

$$\phi_t^{\boldsymbol{f}}(r_0, \theta_0) = (r_0 e^{-t}, \theta_0 + t), \qquad \phi_t^{\boldsymbol{g}}(\rho_0, \psi_0) = (\rho_0 e^{-2t}, \psi_0)$$

and check

$$\boldsymbol{h} \circ \phi_t^{\boldsymbol{f}} = (r_0^2 e^{-2t}, \theta_0 + \ln r_0) = \phi_t^{\boldsymbol{g}} \circ \boldsymbol{h}$$

Example: The dynamical systems

$$\begin{array}{c} \dot{r} = 0 \\ \dot{\theta} = 1 \end{array} \quad \text{and} \quad \begin{array}{c} \dot{r} = 0 \\ \dot{\theta} = r + \sin^2\theta \end{array}$$

are topologically equivalent. This should be obvious because the trajectories are the same and so we can put $\boldsymbol{h}(\boldsymbol{x}) = \boldsymbol{x}$. Then stretch timescale.

Definition 11 ***(Structural Stability)** .The vector field* $\boldsymbol{f}$ *[system* $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ *or flow* $\phi_t^{\boldsymbol{f}}(\boldsymbol{x})$*] is* ***structurally stable*** *if* $\exists \epsilon > 0$ *s.t.* $\boldsymbol{f} + \boldsymbol{\delta}$ *is topologically equivalent to* $\boldsymbol{f}$ $\forall \boldsymbol{\delta}(\boldsymbol{x})$ *with* $|\boldsymbol{\delta}| + \sum_i |\partial \boldsymbol{\delta} / \partial x_i| < \epsilon$.

Examples: System (1) is structurally stable. System (2) is not (since the periodic orbits are destroyed by a small perturbation $\dot{r} \neq 0$).

2 Flows in $\mathbb{R}^2$

2.1 Linearization

In analyzing the behaviour of nonlinear systems the first step is to identify the fixed points. Then near these fixed points, behaviour should approximate linear. In fact near a fixed point $\boldsymbol{x}_0$ s.t. $\boldsymbol{f}(\boldsymbol{x}_0) = 0$, let $\boldsymbol{y} = \boldsymbol{x} - \boldsymbol{x}_0$; then $\dot{\boldsymbol{y}} = \mathsf{A}\boldsymbol{y} + O(|\boldsymbol{y}|^2)$, where $\mathsf{A}_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{\boldsymbol{x}=\boldsymbol{x}_0}$ is the **linearization of $\boldsymbol{f}$ about $\boldsymbol{x}_0$**. The matrix A is also written Df, the **jacobian matrix** of $\boldsymbol{f}$ at $\boldsymbol{x}_0$. We hope that in general the flow near $\boldsymbol{x}_0$ is topologically equivalent to the linearized problem. This is not always true, as shown below.

2.2 Classification of fixed points

Consider general linear system $\dot{\boldsymbol{x}} = \mathsf{A}\boldsymbol{x}$, where A is a constant matrix. We need the eigenvalues $\lambda_{1,2}$ of the matrix, given by $\lambda^2 - \lambda \mathrm{Tr}\mathsf{A} + \mathrm{Det}\mathsf{A} = 0$. This has solutions $\lambda = \frac{1}{2}\mathrm{Tr}\mathsf{A} \pm \sqrt{\frac{1}{4}(\mathrm{Tr}\mathsf{A})^2 - \mathrm{Det}\mathsf{A}}$. We can then classify the roots into classes.

- **Saddle point**($\mathrm{Det}\mathsf{A} < 0$). Roots are real and of opposite sign.
  E.g. $\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 3 \\ 1 & 0 \end{pmatrix}$; (canonical form $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$: $\lambda_1\lambda_2 < 0$.)

- **Node** ($(\mathrm{Tr}\mathsf{A})^2 > 4\mathrm{Det}\mathsf{A} > 0$). Roots are real and either both positive ($\mathrm{Tr}\mathsf{A} > 0$: *unstable, repelling* node), or both negative ($\mathrm{Tr}\mathsf{A} < 0$: *stable, attracting* node).
  E.g. $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & -3 \end{pmatrix}$;(canonical form $\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$: $\lambda_1\lambda_2 > 0$.)

- **Focus (Spiral)** ($(\text{Tr}\mathsf{A})^2 < 4\text{Det}\mathsf{A}$). Roots are complex and either both have positive real part ($\text{Tr}\mathsf{A} > 0$: *unstable, repelling* focus), or both negative real part ($\text{Tr}\mathsf{A} < 0$: *stable, attracting* focus).
  E.g. $\begin{pmatrix} -1 & 1 \\ -1 & -1 \end{pmatrix}$, $\begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}$;(canonical form $\begin{pmatrix} \lambda & \omega \\ -\omega & \lambda \end{pmatrix}$: eigenvalues $\lambda \pm i\omega$.)

Degenerate cases occur when two eigenvalues are equal ($(\text{Tr}\mathsf{A})^2 = 4\text{Det}\mathsf{A} \neq 0$) giving either **Star/Stellar nodes**, e.g. $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or **Improper nodes** e.g. $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

In all these cases the fixed point is **hyperbolic**.

Definition 12 *(Hyperbolic fixed point). A fixed point* $\boldsymbol{x}$ *of a dynamical system is* ***hyperbolic*** *iff all the eigenvalues of the linearization* A *of the system about* $\boldsymbol{x}$ *have non-zero real part.*

This definition holds for higher dimensions too.

Thus the nonhyperbolic cases, which are of great importance in bifurcation theory, are those for which at least one eigenvalue has zero real part. These are of three kinds:

- $\mathsf{A} = 0$. Both eigenvalues are zero.

- $\text{Det}\mathsf{A} = 0$. Here one eigenvalue is zero and we have a line of fixed points. e.g. $\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$

- $\mathrm{Tr}\mathsf{A} = 0$, $\mathrm{Det}\mathsf{A} > 0$ (**Centre**). Here the eigenvalues are $\pm i\omega$ and trajectories are closed curves, e.g. $\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$.

All this can be summarized in a diagram

To find canonical form, find the *eigenvectors* of A and use as basis vectors (possibly generalised if eigenvalues equal), when eigenvalues real. For complex eigenvalues in $\mathbb{R}^2$ we have two complex eigenvectors $\boldsymbol{e}, \boldsymbol{e}^*$ so use $(\mathrm{R}e(\boldsymbol{e}), \mathrm{I}m\boldsymbol{e})$ as a basis. This can help in drawing trajectories. But note classification is independent of basis.

Centres are special cases in context of general flows; but *Hamiltonian systems* have centres generically. Consider $\dot{\boldsymbol{x}} = (H_y, -H_x)$ for some $H = H(x, y)$. At a fixed point $\nabla H = 0$, and the matrix

$$\mathsf{A} = \begin{pmatrix} H_{xy} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix} \Rightarrow \mathrm{Tr}\mathsf{A} = 0.$$

Thus all fixed points are saddles or centres. Clearly also $\dot{\boldsymbol{x}} \cdot \nabla H = 0$, so H is constant on all trajectories.

2.3 Effect of nonlinear terms

For a general nonlinear system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, we start by locating the fixed points, $\boldsymbol{x}_0$, where $\boldsymbol{x}_0 = 0$. Then what does the linearization of the system about the fixed point $\boldsymbol{x}_0$ tell us about the behaviour of the nonlinear system?

We can show (e.g. Glendinning Ch. 4) that if

(i) $\boldsymbol{x}_0$ is hyperbolic; and
(ii)the nonlinear corrections are $O(|\boldsymbol{x} - \boldsymbol{x}_0|^2)$,

Then the two systems are topologically conjugate and that nodes $\Rightarrow$ nodes, foci $\Rightarrow$ $foci$. (Without (ii) nodes and foci cannot be distinguished by topological conjugacy).

We thus discuss separately hyperbolic and non-hyperbolic fixed points.

2.3.1 Stable and Unstable Manifolds

For the *linearized* system we can separate the phase space into different domains corresponding to different behaviours in time.

Definition 13 *(Invariant subspaces). The* ***stable, unstable and centre subspaces*** *of the linearization of* $\boldsymbol{f}$ *at the fixed point* $\boldsymbol{x}_0$ *are the three linear subspaces* E^u, E^s , E^c, *spanned by the subsets of (possibly generalised) eigenvectors of* A *whose eigenvalues have real parts* < 0, > 0, $= 0$ *respectively.*

Note that a hyperbolic fixed point has no centre eigenspace. This concept can be extended simply for hyperbolic fixed points into the nonlinear domain. We suppose that f.p. is at the origin and that $\boldsymbol{f}$ is expandable in a Taylor series. We can write $\dot{\boldsymbol{x}} = \mathsf{A}\boldsymbol{x} + \boldsymbol{F}(\boldsymbol{x})$, where $\boldsymbol{F} = 0(|\boldsymbol{x}|^2)$. We need the **Stable (or Invariant) Manifold Theorem**.

Theorem 3 *(Stable (invariant) Manifold Theorem). Suppose* 0 *is a hyperbolic fixed point of* $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, *and that* E^u, E^s *are the linear unstable and stable subspaces of the linearization of* $\boldsymbol{f}$ *about* 0. *Then* $\exists$ ***local stable and unstable manifolds*** $W^u_{loc}(0)$, $W^s_{loc}(0)$, *which have the same dimension as* E^u, E^s *and are tangent to* E^u, E^s *at* 0, *such that for* $\boldsymbol{x} \neq 0$ *but in a sufficiently small neighbourhood of* 0,

$$\begin{aligned} W^u_{loc} &= \{\boldsymbol{x} \ : \ \phi_t(\boldsymbol{x}) \to 0 \ \textit{as } t \to -\infty\} \\ W^s_{loc} &= \{\boldsymbol{x} \ : \ \phi_t(\boldsymbol{x}) \to 0 \ \textit{as } t \to +\infty\} \end{aligned}$$

Proof: rather involved; see Glendinning, p.96. The trick is to produce a near identity change of coordinates. Suppose that $\boldsymbol{x}$, $\boldsymbol{y}$ span the unstable(stable) subspaces $E^u(E^s)$, so the linearized stable manifold is $\boldsymbol{x} = 0$. We look for the (e.g.) stable manifold in the form $\boldsymbol{x} = \boldsymbol{S}(\boldsymbol{y})$; then make a change of variable $\boldsymbol{\xi} = \boldsymbol{x} - \boldsymbol{S}(\boldsymbol{y})$ so that the transformed equation has $\boldsymbol{\xi} = 0$ an an invariant manifold. The function $\boldsymbol{S}(\boldsymbol{y})$ can be expanded as a power series, and the idea is to check that the expansion can be performed to all orders, giving a finite radius of convergence.

The local stable (unstable) manifold can be extended to a *global invariant manifold* $W^s(W^u)$ by following the flow backwards (forwards) in time from a points in W^s_{loc} (W^u_{loc}).

It is easy to find approximations to the stable and unstable manifolds of a saddle point in $\mathbb{R}^2$. The stable(say) manifold must tend to the origin and be tangent to the stable subspace E^s (i.e. to the eigenvector corresponding to the negative eigenvalue. (It is often easiest though not necessary to change to coordinates such that $x = 0$ or $y = 0$ is tangent to the manifold). Then for example if we want to find the manifold (for 2D flows just a trajectory) that is tangent to $y = 0$ at the origin for the system $\dot{x} = f(x, y)$, $\dot{y} = g(x, y)$, write $y = p(x)$; then

$$g(x, p(x)) = \dot{y} = p'(x)\dot{x} = p'(x)f(x, p(x)),$$

which gives a nonlinear ODE for $p(x)$. In general this cannot be solved exactly, but we can find a (locally convergent) series expansion in the form $p(x) = a_2x^2 + a_3x^3 + \ldots$, and solve term by term.

Example 10 *Consider* $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x \\ -y + x^2 \end{pmatrix}$. *This can be solved exactly to give* $x = x_0 e^t$, $y = \frac{1}{3}x_0^2 \mathrm{e}^{2t} + (y_0 - \frac{1}{3}x_0^2)\mathrm{e}^{-t}$ *or* $y(x) = \frac{1}{3}x^2 + (y_0 - \frac{1}{3}x_0^2)x_0 x^{-1}$. *Two obvious invariant curves are* $x = 0$ *and* $y = \frac{1}{3}x^2$, *and* $x = 0$ *is clearly the stable manifold. The linearization about* 0 *gives the matrix* $\mathsf{A} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ *and so the unstable manifold must be tangent to* $y = 0$; $y = \frac{1}{3}x^2$ *fits the bill. To find constructively write* $y(x) = a_2 x^2 + a_3 x^3 + \ldots$. *Then*

$$\frac{dy}{dt} = \dot{x}\frac{dy}{dx} = (2a_2 x + 3a_3 x^2 + \ldots)x = -a_2 x^2 - a_3 x^3 + \ldots + x^2$$

Equating coefficients, find $a_2 = \frac{1}{3}$, $a_3 = 0$, *etc.*

Example 11 *Now* $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x - xy \\ -y + x^2 \end{pmatrix}$; *there is no simple form for the unstable manifold (stable manifold is still* $x = 0$*). The unstable manifold has the form* $y = ax^2 + bx^3 + cx^4 + \ldots$, *where [exercise]* $a = \frac{1}{3}$, $b = 0$, $c = \frac{2}{45}$, *etc. Note that this infinite series (in powers of* x^2*) has a finite radius of convergence since the unstable manifold of the origin is attracted to a stable focus at* $(1, 1)$.

2.3.2 Nonlinear terms for non-hyperbolic cases

We now suppose that there is at least one eigenvalue on the imaginary axis. Concentrate on $\mathbb{R}^2$, generalization not difficult. There are two possibilities:

(i) A has eigenvalues $\pm i\omega$. The linear system is a centre. The nonlinear system have different forms for different r.h.s.'s

- Stable focus: $\begin{pmatrix} -y - x^3 \\ x - y^3 \end{pmatrix}$

- Unstable focus: $\begin{pmatrix} -y + x^3 \\ x + y^3 \end{pmatrix}$

- Nonlinear centre: $\begin{pmatrix} -y - 2x^2 y \\ x + 2y^2 x \end{pmatrix}$

(ii) A has one zero eigenvalue, other e.v.non-zero, e.g. (a) $\dot{x} = x^2,\ \dot{y} = -y$ [Saddle-node], (b) $\dot{x} = x^3,\ \dot{y} = -y$ [Nonlinear Saddle].

(iii) Two zero eigenvalues. Here almost anything is possible. Change to polar coords. Find lines as $r \to 0$ on which $\dot{\theta} = 0$. Between each of these lines can have three different types of behaviour. (See diagram).

2.4 Sketching phase portraits

This often involves some good luck and good judgement! Nonetheless there are some guidelines which if followed will give a good chance of success. The general procedure is as follows:

- 1. Find the fixed points, and find any obvious invariant lines e.g.$x = 0$ when $x = xh(x, y)$ etc.
- 2. Calculate the jacobian and hence find the type of fixed point. (Accurate calculation of eigenvalues etc. for nodes may not be needed for sufficiently simple systems - just find the type.) Do find eigenvectors for saddles.
- 3. If fixed points non-hyperbolic get local picture by considering nonlinear terms.
- 4. If still puzzled, find nullclines, where x or y (or r or θ) are zero.
- 5. Construct global picture by joining up local trajectories near fixed points (especially saddle separatrices) and put in arrows.
- 6. Use results of Ch. 4 to decide whether there are periodic orbits.

Example 12 *(worked example). Consider* $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(1-y) \\ -y+x^2 \end{pmatrix}$. *Jacobian* $\mathsf{A} = \begin{pmatrix} 1-y & -x \\ 2x & -1 \end{pmatrix}$.

Fixed points at $(0, 0)$ *(saddle) and* $(\pm 1, 1)$; $\mathsf{A} = \begin{pmatrix} 0 & \mp 1 \\ \pm 2 & -1 \end{pmatrix}$. $\mathrm{Tr}\mathsf{A}^2 = 1 < \mathrm{Det}\mathsf{A}$ *so stable foci.*

$x = 0$ *is a trajectory,* $\dot{x} = 0$ *on* $y = 1$ *and* $\dot{y} \lesseqgtr 0$ *when* $y \gtreqless x^2$.

3 Stability

3.1 Definitions of stability

If we find a fixed point, or more generally an invariant set, of an o.d.e. we want to know what happens to the system under small perturbations. We also want to know which invariant sets will be approached at large times. If in some sense the solution stays "nearby", or the set is approached after long times, then we call the set stable. There are several differing definitions of stability; different texts are not consistent. I shall define stability of whole invariant sets (and not just of points in those sets). This shortens the discussion.

Consider an invariant set Λ in a general (autonomous) dynamical system described by a flow ϕ_t. t (This could be a fixed point, periodic orbit, torus etc.) We need a definition of points near the set Λ:

Definition 14 *(Neighbourhood of a set Λ). For $\delta > 0$ the **neighbourhood** $N_\delta(\Lambda) = \{\boldsymbol{x} : \exists \boldsymbol{y} \in \Lambda \text{ s.t. } |\boldsymbol{x} - \boldsymbol{y}| < \delta\}$*

We also need to define the concept of a flow trajectory tending to Λ.

Definition 15 *(flow tending to Λ). The flow $\phi_t(\boldsymbol{x}) \to \Lambda$ iff $\min_{\boldsymbol{y}\in\Lambda} |\phi_t(\boldsymbol{x}) - \boldsymbol{y}| \to 0$ as $t \to \infty$*

Definition 16 *(Lyapunov stability)[LS]. The set Λ is **Lyapunov stable** if $\forall \epsilon > 0 \, \exists \delta > 0$ s.t. $\boldsymbol{x} \in N_\delta(\Lambda) \Rightarrow \phi_t(\boldsymbol{x}) \in N_\epsilon(\Lambda) \; \forall t \geq 0$. **("start near, stay near")**.*

Definition 17 *(Quasi-asymptotic stability)[QAS]. The set Λ is **quasi-asymptotically stable** if $\exists \delta > 0$ s.t.$\boldsymbol{x} \in N_\delta(\Lambda) \Rightarrow \phi_t(\boldsymbol{x}) \to \Lambda$ as $t \to \infty$. **("get close eventually")**.*

Definition 18 *(Asymptotic stability)[AS]. The set Λ is* ***asymptotically stable*** *if it is both Lyapunov stable and quasi-asymptotically stable.*

Example 13 *(LS but not QAS).* $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$.
All limit sets are circles or the origin.

Example 14 *(QAS but not LS).* $\begin{pmatrix} \dot{r} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} r(1-r^2) \\ \sin^2 \frac{\theta}{2} \end{pmatrix}$.
Point $r = 1, \theta = 0$ is a saddle-node.

Example 15 $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - \epsilon x \\ -2\epsilon y \end{pmatrix}$ *$(\epsilon > 0)$. Here $y = y_0 e^{-2\epsilon t}$, $x = x_0 e^{-\epsilon t} + y_0 \epsilon^{-1}(e^{-\epsilon t} - e^{-2\epsilon t})$. Thus for $t \geq 0$, $|y| \leq |y_0|$, $|x| \leq |x_0| + \frac{1}{4}\epsilon^{-1}|y_0|$, and so $x^2 + y^2 \leq (x_0^2 + y_0^2)(1 + \epsilon^{-1} + \epsilon^{-2}/16)$. This proves Lyapunov stability. Furthermore the solution clearly tends to the origin as $t \to \infty$.*

This example is instructive because for ϵ sufficiently small the solution can grow to large values before eventually decaying. To require that the norm of the solution decays monotonically is a stronger result, only applicable to a small number of problems.

If an invariant set is not LS or QAS we say it is **unstable** (or according to some books, **nonstable**).

There may be more than one asymptotically stable limit set. In that case we want to know what parts of the phase space lead to which limit sets being approached. Then we define the **basin of attraction**(or domain of stability):

Definition 19 *If Λ is an asymptotically stable invariant set the* ***basin of attraction of*** $\boldsymbol{\Lambda}$*,* $\mathcal{B}(\Lambda) \equiv \{\boldsymbol{x} : \ \phi_t(\boldsymbol{x}) \to \Lambda \text{ as } t \to \infty\}$*. If* $\mathcal{B}(\Lambda) = \mathbb{R}^n$ *then* Λ *is* ***globally attracting****(or globally stable). Note that* $\mathcal{B}(\Lambda)$ *is an open set.*

When there are many fixed points the basin of attraction can be quite complicated. (See handout).

When Λ is an isolated fixed point ($\boldsymbol{x}_0$, say) we can investigate its stability by **linearizing** the system about $\boldsymbol{x}_0$(see previous chapter).

Theorem 4 *(Stability of hyperbolic fixed points). If* 0 *is a hyperbolic sink then it is asymptotically stable. If* 0 *is a hyperbolic fixed point with at least one eigenvalue with* $\Re\lambda > 0$*, then it is unstable.*

3.2 Lyapunov functions

We can prove a lot about stability of a fixed point (which can be taken to be at the origin) if we can find a positive function $\mathcal{V}$ of the independent variables that decreases monotonically under the flow ϕ_t. Then under certain reasonable conditions we can show that $\mathcal{V} \to 0$ so that the appropriately defined modulus of the solution similarly tends to zero. This is a **Lyapunov function**, defined precisely by

Definition 20 *(Lyapunov function). Let E be a closed connected region of $\mathbb{R}^n$ containing the origin. A function $\mathcal{V} : \mathbb{R}^n \to \mathbb{R}$ which is differentiable except perhaps at the origin is a* ***Lyapunov function*** *for a flow ϕ if (i) $\mathcal{V}(0) = 0$, (ii) $\mathcal{V}$ is* ***positive definite*** *($\mathcal{V}(\boldsymbol{x}) > 0$ when $0 \neq \boldsymbol{x}$), and if also (iii) $\mathcal{V}(\phi_t(\boldsymbol{x})) \leq \mathcal{V}(\boldsymbol{x})$ $\forall \boldsymbol{x} \in E$ (or equivalently if $\dot{\mathcal{V}} \leq 0$ on trajectories).*

Then we have the following theorems:

Theorem 5 *(Lyapunov's First Theorem [L1]). Suppose that a dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ has a fixed point at the origin. If a Lyapunov function exists,as defined above, then the origin is Lyapunov stable.*

Theorem 6 *(Lyapunov's Second Theorem [L2]). If in addition $\dot{\mathcal{V}} < 0$ for $\boldsymbol{x} \neq 0$ then $\mathcal{V}$ is a* ***Strict Lyapunov function*** *and the origin is asymptotically stable.*

Proof of First Theorem: We want to show that for any sufficiently small neighbourhood U of the origin, there is a neighbourhood V s.t. if $\boldsymbol{x}_0 \in V$, $\phi_t(\boldsymbol{x}_0) \in U$ for all positive t. We start by choosing δ sufficiently small that $|\boldsymbol{x}| < \delta \Rightarrow \boldsymbol{x} \in U$. Let $\alpha = \min\{\mathcal{V} : |\boldsymbol{x}| = \delta\}$. Clearly $\alpha > 0$ from the definition of $\mathcal{V}$. Now consider the set $U_1 = \{\boldsymbol{x} : \mathcal{V}(\boldsymbol{x}) < \alpha \text{ and } |\boldsymbol{x}| \leq \delta\}$. Then since for $\boldsymbol{x}_0 \in U_1$, $\mathcal{V}(\boldsymbol{x}_0) < \alpha$ and $\mathcal{V}$ does not increase along trajectories, $|\phi_t(\boldsymbol{x}_0)|$ can never reach δ, as if this happened $\mathcal{V}$ would be $\geq \alpha$. Thus clearly $\boldsymbol{x} \in U_1$ at $t = 0 \Rightarrow |\boldsymbol{x}(t)| < \delta \Rightarrow \boldsymbol{x}(t) \in U \forall t \geq 0$, as required.

Proof of Second Theorem: [L1] implies that $\boldsymbol{x}$ remains in the domain U. For any initial point $\boldsymbol{x}_0 \neq 0$, $\mathcal{V}(\boldsymbol{x}_0) > 0$ and $\dot{\mathcal{V}} < 0$ along trajectories. Thus $\mathcal{V}(\phi_t(\boldsymbol{x}_0))$ decreases montonically and is bounded below by 0. Then $\mathcal{V} \to \alpha \geq 0$; suppose $\alpha > 0$. Then $|\boldsymbol{x}|$ is bounded away from zero (by continuity of $\mathcal{V}$), and so $\mathcal{W} \equiv \dot{\mathcal{V}} < -b < 0$. Thus $\mathcal{V}(\phi_t(\boldsymbol{x}_0)) < \mathcal{V}(\boldsymbol{x}_0) - bt$ and so is certainly negative after a finite time. Thus there is a contradiction, and so $\alpha = 0$. Hence $\mathcal{V} \to 0$ as $t \to \infty$ and so $|\boldsymbol{x}| \to 0$ (again by continuity). This proves asymptotic stability.

To prove results about instability just reverse the sense of time. Sometimes we can demonstrate asymptotic stability even when $\mathcal{V}$ is not a strict Lyapunov function. For this need another theorem, **La Salle's Invariance Principle**:

Theorem 7 *(La Salle's Invariance Principle). If $\mathcal{V}$ is a Lyapunov function for a flow ϕ then $\forall \boldsymbol{x}\, \exists c$ s.t. $\omega(\boldsymbol{x}) \in M_c \equiv \{\boldsymbol{x} : \; \mathcal{V}(\phi_t(\boldsymbol{x})) = c \;\forall t \geq 0\}$.(Or, $\phi_t(\boldsymbol{x}) \to$ an <u>invariant</u> subset of the set $\{\boldsymbol{y} : \dot{\mathcal{V}}(\boldsymbol{y}) = 0\}$.)*

Proof : choose a point $\boldsymbol{x}$ and let $c = \inf_{t\geq 0} \mathcal{V}(\phi_t(\boldsymbol{x}))$. If $c = -\infty$ then $\phi_t(\boldsymbol{x}) \to \infty$ and $\omega(\boldsymbol{x})$ is empty. So suppose $\phi_t(\boldsymbol{x})$ remains finite and that $\omega(\boldsymbol{x})$ is not empty. Then if $\boldsymbol{y} \in \omega(\boldsymbol{x})$, $\exists$ a sequence of times t_n s.t. $\phi_{t_n}(\boldsymbol{x}) \to \boldsymbol{y}$ as $n \to \infty$, then by continuity of $\mathcal{V}$ and since $\dot{\mathcal{V}} \leq 0$ on trajectories we have $\mathcal{V}(\boldsymbol{y}) = c$. We need to prove that $\boldsymbol{y} \in M_c$ (i.e. that $\mathcal{V}(\boldsymbol{y}) = c$ for $t \geq 0$). Suppose to the contrary that $\exists s$ s.t. $\mathcal{V}(\phi_s(\boldsymbol{y})) < c$. Thus for all $\boldsymbol{z}$ sufficiently close to $\boldsymbol{y}$ we have $\mathcal{V}(\phi_s(\boldsymbol{z})) < c$. But if $\boldsymbol{z} = \phi_{t_n}(\boldsymbol{x})$ for sufficiently large n we have $\phi_{t_n+s}(\boldsymbol{x}) < c$, which is a contradiction. This proves the theorem.

As a corollary we note that if $\mathcal{V}$ is a Lyapunov function on a bounded domain D and the only invariant subset of $\{\dot{\mathcal{V}} = 0\}$ is the origin then the origin is asymptotically stable.

Examples of the use of the Lyapunov theorems.

Example 16 *(Finding the basin of attraction). Consider* $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} -x + xy^2 \\ -2y + yx^2 \end{pmatrix}$. *We can ask: what is the best condition on $\boldsymbol{x}$ which guarantees that $\boldsymbol{x} \to 0$ as $t \to \infty$? Consider*

$\mathcal{V}(x,y) = \frac{1}{2}(x^2 + b^2y^2)$ for constant b. Then $\dot{\mathcal{V}} = -(x^2 + 2b^2y^2) + (1+b^2)x^2y^2$ We can then show easily that
$\dot{\mathcal{V}} < 0$ if $\mathcal{V} < (3+2\sqrt{2})b^2/2(1+b^2)$ [Check by setting $z = by$ and using polars for (x,z)]. The domain of attraction of the origin certainly includes the union of all these sets (see handout).

Example 17 *(Damped pendulum). Here* $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -ky - \sin x \end{pmatrix}$ *with $k > 0$. We can choose $\mathcal{V} = \frac{1}{2}y^2 + 1 - \cos x$; clearly $\mathcal{V}$ is positive definite provided we identify x and $x + 2\pi$, and $\dot{\mathcal{V}} = -ky^2 \leq 0$. So certainly the origin is Lyapunov stable by* [L1]. *But we cannot use* [L2] *since $\dot{\mathcal{V}}$ is not negative definite. Nonetheless from La Salle's principle we see that the set M_c is contained in the set of complete orbits satisfying $y = 0$. The only such orbits are $(0,0)$ $(c = 0)$ and $(\pi, 0)$ $(c = 2)$ So these points are the only possible members of $\omega(\boldsymbol{x})$. Since the origin is Lyapunov stable we conclude that for all points $\boldsymbol{x}$ s.t. $\mathcal{V} < 2$ the only member of $\omega(\boldsymbol{x})$ is the origin and so this point is asymptotically stable.*

Special cases are *gradient flows.*

Definition 21 *A system is called a* ***gradient system*** *or* ***gradient flow*** *if we can write $\dot{\boldsymbol{x}} = -\nabla V(\boldsymbol{x})$.*

In this case we have $\dot{V} = -|\nabla V|^2 \leq 0$, with $\dot{V} < 0$ except at the fixed points which have $|\nabla V| = 0$. Thus we can use La Salle's principle to show that $\Omega = \{\omega(\boldsymbol{x}) : \boldsymbol{x} \in E\}$ consists only of the fixed points. Note that this does NOT mean that all the fixed points are asymptotically stable (see diagram). These ideas can be extended to more general systems of the form $\dot{\boldsymbol{x}} = -h\nabla(V)$, where $h(\boldsymbol{x})$ is a strictly positive continuously differentiable function. [Proof: exercise].

3.3 Bounding functions

Even when we cannot find a Lyapunov function in the exact sense, we can sometimes find positive definite functions $\mathcal{V}$ s.t. $\dot{\mathcal{V}} < 0$ *outside* some neighbourhood of the origin. We call these **Bounding functions**. They are used to show that $\boldsymbol{x}$ remains in some neighbourhood of the origin.

Theorem 8 *Let $\mathcal{V} : E \subseteq \mathbb{R}^n \to \mathbb{R}$ be a positive definite continuously differentiable function and let D be a region s.t. $\dot{\mathcal{V}}(\boldsymbol{x}) < -\delta$ $(\delta > 0)$ if $\boldsymbol{x} \notin D$, and let $\alpha = sup\{\mathcal{V}(\boldsymbol{x}) : \boldsymbol{x}\, on\ boundary\ of\ D\}$. Then all orbits eventually enter and remain inside the set $\mathcal{V}_\alpha = \{\boldsymbol{x} : \mathcal{V}(\boldsymbol{x}) \leq \alpha\}$.*

Proof: exercise (see diagram).

4 Existence and stability of periodic orbits in $\mathbb{R}^2$

Example 18 *Damped pendulum with torque Consider the system* $\ddot{\theta} + k\dot{\theta} + \sin\theta = F$, $k > 0$, $F > 0$. *We would like to know whether there is a periodic orbit of this equation. We can find a bounding function of the form* $\mathcal{V} = \frac{1}{2}p^2 + 1 - \cos\theta$ $(p = \dot{\theta})$. *Then* $\dot{\mathcal{V}} = p\dot{p} + \dot{\theta}\sin\theta = p(F - kp)$. *Thus* $\dot{\mathcal{V}} < 0$ *unless* $0 < p < F/k$. *Maximum value of* $\mathcal{V}$ *on boundary of this domain is* $\mathcal{V}_{max} = \frac{1}{2}(\frac{F}{k})^2 + 2$. *By previous result on bounding functions we see that all orbits eventually enter and remain in the region* $\mathcal{V} < \mathcal{V}_{max}$.

What happens within this region? We can look for fixed points: these are at $p = 0$, $\sin\theta = F$. *So there are 2 f.p.'s if* $F < 1$, *no f.p.'s if* $F > 1$. *In the first case one fixed point is stable (node or focus depending on* k*) and the other is a saddle. In the second case what can happen? Either there is a closed orbit or ?possibly? a space filling curve. There are some nontrivial theorems that we can use to answer this.*

4.1 The Poincaré Index

Closed curves can be distinguished by the number of rotations of the vector field $\boldsymbol{f}$ as the curve is traversed. This property of a curve in a vector field is very useful in understanding the phase portrait.

Theorem 9 *(Poincaré Index). Consider a vector field* $\boldsymbol{f} = (f_1, f_2)$. *At any point the direction* *of* $\boldsymbol{f}$ *is given by* $\psi = \tan^{-1}(f_2/f_1)$ *(with the usual conventions). Now let* $\boldsymbol{x}$ *traverse a closed curve* $\mathcal{C}$*.* ψ *will increase by some multiple (possibly negative) or* 2π*. This multiple is called the* ***Poincaré Index*** $I_{\mathcal{C}}$ *of* $\mathcal{C}$*.*

This index can be put in integral form. We have

$$I_{\mathcal{C}} = \frac{1}{2\pi}\oint_{\mathcal{C}} d\psi = \frac{1}{2\pi}\oint_{\mathcal{C}} d\tan^{-1}\left(\frac{f_2}{f_1}\right) = \frac{1}{2\pi}\oint_{\mathcal{C}} \frac{f_1 df_2 - f_2 df_1}{f_1^2 + f_2^2}$$

In fact the index is most easily worked out by hand. There are several results that are easily proved about Poincaré indices that makes them easier to calculate.

1. **The index takes only integer values, and is continuous when the vector field has no zeroes.** It therefore is the same for two curves which can be deformed into each other without crossing any fixed point.

2. **The index of any curve not enclosing any fixed point is zero.** This is because it can be shrunk to zero size.

3. **The index of a curve enclosing a number of fixed points is the sum of the indices for curves enclosing the fixed points individually.** This is because the curve can be deformed into small curves surrounding each fixed point together with connecting lines along which the integral cancels.

4. **The index of a curve for $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ is the same as that of the system $\dot{\boldsymbol{x}} = -\boldsymbol{f}(\boldsymbol{x})$.** Proof: Consider effect on integral representation of the change $\boldsymbol{f} \to -\boldsymbol{f}$.

5. **The index of a periodic orbit is $+1$.** The vector field is tangent to the orbit at every point.

6. **The index of a saddle is -1, and of a node or focus $+1$.** By inspection, or by noting that in complex notation a saddle can be written, in suitable coordinates as $\dot{x} = x$, $\dot{y} = -y$ or $\dot{z} = \bar{z}$. For a node or focus, curve can be found such that trajectories cross in same direction.

7. **Indices of more complicated, non hyperbolic points can be found by adding the indices for the simpler fixed points that may appear under perturbation.** This is because a small change in the system does not change the index round a curve where the vector field is smooth. e.g. index of a saddle-node $\dot{x} = x^2$, $\dot{y} = -y$ is zero, since indices of saddle and node cancel.

The most important non-trivial result that can be proved is that any periodic orbit contains at least one fixed point. In fact the total number of nodes and foci must exceed the total number of saddles by one. Proof: simple exercise.

4.2 Poincaré-Bendixson Theorem

This remarkable result, which only holds in $\mathbb{R}^2$, is very useful for proving the existence of periodic orbits.

Theorem 10 *(Poincaré-Bendixson). Consider the system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, $\boldsymbol{x} \in \mathbb{R}^2$, and suppose that $\boldsymbol{f}$ is continuously differentiable. If the forward orbit $\mathcal{O}^+(\boldsymbol{x})$ remains in a compact (closed and bounded) set containing no fixed points then $\omega(\boldsymbol{x})$ contains a periodic orbit.*

We can apply this directly to the pendulum equation with $F > 1$ to show that there is at least one (stable) periodic orbit. However we cannot rule out multiple periodic orbits.

Proof: not trivial!! We first note that since $\mathcal{O}^+(\boldsymbol{x})$ remains in a compact set, then $\omega(\boldsymbol{x})$ is non-empty. Recall that $\omega(\boldsymbol{x})$ is invariant under $\boldsymbol{f}$. Choose $\boldsymbol{y} \in \omega(\boldsymbol{x})$, and pick $\boldsymbol{z} \in \omega(\boldsymbol{y})$. Since $\boldsymbol{z}$ is not a fixed point, will show first that $\mathcal{O}^+(\boldsymbol{y})$ is a periodic orbit. Pick a curve Σ transverse to flow. Then by continuity of $\boldsymbol{f}$ trajectories cross Σ in the same direction everywhere in a neighbourhood of $\boldsymbol{z}$. Since $\boldsymbol{z} \in \omega(\boldsymbol{y})$, $\mathcal{O}^+(\boldsymbol{y})$ comes arbitrarily close to $\boldsymbol{z}$ as $t \to \infty$, and so makes intersections with Σ arbitrarily close to $\boldsymbol{z}$. If in fact these are all at the same point, we have a periodic orbit.

But suppose not. Then by uniqueness of the flow all intersections with Σ must be on the same side of $\boldsymbol{z}$, and *distinct*. This latter follows from uniqueness of flow, the former from the Jordan curve lemma (see diagrams). If $\boldsymbol{y}_1 \neq \boldsymbol{y}_2$ (first two intersections distinct) then the orbit $\mathcal{O}^+(\boldsymbol{x})$ cannot return to the nbd. of $\boldsymbol{y}_i$ (again,see picture + use Jordan curve lemma). But $\boldsymbol{y}_i \in \omega(\boldsymbol{x})$, so we have a contradiction. Thus in fact $\boldsymbol{y}_1 = \boldsymbol{y}_2$ and $\mathcal{O}^+(\boldsymbol{y})$ *is* a periodic orbit, and since $\mathcal{O}^+(\boldsymbol{y}) \subset \omega(\boldsymbol{x})$ we have that $\omega(\boldsymbol{x})$ contains a periodic orbit. Now in fact $\mathcal{O}^+(\boldsymbol{y}) = \omega(\boldsymbol{x})$. To see this take a transverse section Σ through the orbit at $\boldsymbol{y}$, say, and look at intersections of close parts of $\phi_t(\boldsymbol{x})$ with Σ. If an intersection is at $\boldsymbol{y}$ then $\boldsymbol{x}$ is on the orbit and $\omega(\boldsymbol{x}) = \mathcal{O}^+(\boldsymbol{y})$. Otherwise as before we have a monotonic sequence of intersections tending to $\boldsymbol{y}$ and so again $\omega(\boldsymbol{x}) = \mathcal{O}^+(\boldsymbol{y})$.

Example 19 *Consider the system*
$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x - y + 2x^2 + axy - x(x^2+y^2) \\ y + x + 2xy - ax^2 - y(x^2+y^2) \end{pmatrix}$. *In polar coordinates we get* $\dot{r} = r + 2r^2\cos\theta - r^3$, $\dot{\theta} = 1 - ar\cos\theta$. *Then* $\dot{r} > 0$ *for* $r < \sqrt{2} - 1$, *and* $\dot{r} < 0$ *for* $r > \sqrt{2} + 1$. *Thus the trajectories enter the annulus* $\sqrt{2} - 1 \leq r \leq \sqrt{2} + 1$. *For any fixed points we have* $1 + 2r\cos\theta - r^2 = 0$, $1 - ar\cos\theta = 0$. *Hence* $x = 1/a$, $r^2 = 1 + 2/a$. *So there are no fixed points in the annulus, and so no periodic solutions, if* $1/a > 1 + \sqrt{2}$ *or if* $1/a < 1 - \sqrt{2}$.

4.3 Dulac's criterion and the divergence test

Suppose that $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ has a periodic orbit $\mathcal{C}$. Then $\boldsymbol{f}$ is tangent to $\mathcal{C}$ at every point and since there are no fixed points on $\mathcal{C}$ we have that $\rho(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}) \cdot \boldsymbol{n} = 0$ everywhere on $\mathcal{C}$, where ρ is a C^1 non-negative function and $\boldsymbol{n}$ is the unit outward normal. Then

$$\oint_{\mathcal{C}} \rho \boldsymbol{f} \cdot \boldsymbol{n} d\ell = \int_A \nabla \cdot (\rho \boldsymbol{f}) dA = 0$$

Thus unless ρ and $\boldsymbol{f}$ are such that $\nabla \cdot (\rho \boldsymbol{f})$ is not all of one sign within the periodic orbit, we have a contradiction. This is is a special case of

Theorem 11 *(Dulac's negative criterion). Consider a dynamical system* $\dot{\boldsymbol{x}} = \boldsymbol{f}$, $\boldsymbol{x} \in \mathbb{R}^n$ *and a non negative scalar function* $\rho(\boldsymbol{x})$, *so that* $\nabla \cdot (\rho \boldsymbol{f}) < 0$ *everywhere in some domain* $E \subseteq \mathbb{R}^n$, *or if the same expression is > 0 everywhere in E. Then there are no invariant sets of finite volume V lying wholly within E.*

Proof: suppose there is such a set V, and consider ρ as a "density"; then $\int_V \rho dV$ is the total "mass" within V and since V is invariant this is independent of time. Thus the total "mass flux" $\int_{\partial V} \rho \boldsymbol{f} \cdot \boldsymbol{n} dS$ out of V is zero, and this equals $\int_V \nabla \cdot (\rho \boldsymbol{f}) dV$ from the divergence theorem. Thus we have a contradiction.

Often it is only necessary to take $\rho = 1$. If conditions of the theorem satisfied in $\mathbb{R}^2$ then there are no periodic orbits in E, if in $\mathbb{R}^3$ there are no invariant 2-tori, etc.

Example 20 *(Lorenz system). Here we are in $\mathbb{R}^3$ and equations are*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} \sigma y - \sigma x \\ rx - y - zx \\ -bz + xy \end{pmatrix} .$$

Hence $\nabla \cdot \boldsymbol{f} = -\sigma - 1 - b < 0$. Thus this system has no invariant sets (including tori) with non-zero volume. If there is a strange attractor it must have zero volume (typically it will be a fractal set). Note that periodic orbits (of zero volume) are not excluded.

Example 21 *(Damped pendulum with torque). We have seen using Poincaré-Bendixson that when $F > 1$ there are no fixed points and that there must be a periodic orbit encircling the cylinder. IN this case $\nabla \cdot \boldsymbol{f} = -k < 0$ and so there are no periodic orbits* ***not*** *encircling the cylinder. We cannot rule out orbits that do encircle the cylinder as they have no 'interior' and so theorem does not apply. But suppose there are two periodic orbits (both necessarily encircling the cylinder). Then the region between them on the cylinder surface must be invariant, which is not allowed by a simple extension of the criterion above. Thus we have a unique stable periodic orbit of rotational type when $F > 1$.*

Example 22 *(Predator-Prey equations). These take the general form*

$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x(A - a_1 x + b_1 y) \\ y(B - a_2 y + b_2 x) \end{pmatrix}$, *where* $a_1,\, a_2 > 0$. *The lines* $x = 0,\ y = 0$ *are invariant. Are there periodic orbits in the first quadrant? Consider* $\rho = (xy)^{-1}$. *Then* $\nabla \cdot \rho \boldsymbol{f} = \frac{\partial}{\partial x}(y^{-1}(A - a_1 x + b_1 y) + \frac{\partial}{\partial y}(x^{-1}(B - a_2 y + b_2 x) = -a_1 y^{-1} - a_2 x^{-1} < 0$. *So no periodic orbits in this case. When* $a_1 = a_2 = 0$ *we have an infinite number of periodic orbits, which are contours of* $U(x, y) = A \ln y - B \ln x + b_1 y - b_2 x$, *provided these are closed curves [when is this true??]*

Another result that is sometimes useful relies on the fact that if there is a periodic orbit the vector field $\boldsymbol{f}$ is everywhere tangent to the orbit and is never zero.

Theorem 12 *(Gradient criterion). Consider a dynamical system* $\dot{\boldsymbol{x}} = \boldsymbol{f}$ *where* $\boldsymbol{f}$ *is defined throughout a simply connected domain* $E \subset \mathbb{R}^2$. *If there exists a positive function* $\rho(\boldsymbol{x})$ *such that* $\rho \boldsymbol{f} = \nabla \psi$ *for some single valued function* ψ, *then there are no periodic orbits lying entirely within* E.

Proof: if $\rho \boldsymbol{f} = \nabla \psi$ then for a periodic orbit $\mathcal{C}$, $\oint_{\mathcal{C}} \rho \boldsymbol{f} \cdot d\ell = \oint_{\mathcal{C}} \nabla \psi \cdot d\ell = 0$. But this is impossible as $\rho \boldsymbol{f} \cdot d\ell$ has the same sign everywhere on $\mathcal{C}$.

Example 23 *Consider the system* $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 2x + xy^2 - y^3 \\ -2y - xy^2 + x^3 \end{pmatrix}$. *Hard to apply Dulac. But in fact* $\mathrm{e}^{xy} \boldsymbol{f} = \nabla(\mathrm{e}^{xy}(x^2 - y^2))$ *and so there are no periodic orbits.*

4.4 Near-Hamiltonian flows

Many systems of importance have a Hamiltonian structure; that is they may be written in the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} H_y \\ -H_x \end{pmatrix}$$

Then it is easy to see that $\frac{dH}{dt} = \dot{x}H_x + \dot{y}H_y = 0$ so that the curves $H = \text{const.}$ are invariant. If these curves are closed then they are periodic orbits, but not limit cycles since they are not isolated. The jacobian of $\boldsymbol{f}$ at the fixed point is $\begin{pmatrix} H_{xy} & H_{yy} \\ -H_{xx} & -H_{xy} \end{pmatrix}$. Thus the trace is always zero and fixed points are either saddles or (nonlinear) centres.

Example 24 *Consider $\ddot{x} + x - x^2$. Writing $y = \dot{x}$, we have an equation in hamiltonian form, with $H = \frac{1}{2}(y^2 + x^2) - \frac{1}{3}x^3$. There are two fixed points, at $y = 0$, $x = 0, 1$, which are a centre and a saddle respectively. Using symmetry about the x-axis we can construct the phase portrait.*

If a system can be written in Hamiltonian form with perturbations, we can find conditions for periodic orbits. Suppose we have the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} H_y + g_1(x, y) \\ -H_x + g_2(x, y) \end{pmatrix} \tag{2}$$

then we can see that

$$\frac{dH}{dt} = g_2\dot{x} - g_1\dot{y} \ . \tag{3}$$

If there is a closed orbit $\mathcal{C}$, we have $\oint_{\mathcal{C}} (g_1 H_x + g_2 H_y) dt = \oint_{\mathcal{C}} g_2 dx - g_1 dy = 0$. If we can show that these line integrals cannot vanish, we have shown that there can be no periodic orbits.

Example 25 *Consider $\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y \\ -x + x^2 + \epsilon y(a - x) \end{pmatrix}$ ($\epsilon > 0$). Choose the same value of H as in the previous example; then for a periodic orbit $\mathcal{C}$ we must have $\oint_{\mathcal{C}} y^2(a - x)\, dt = 0$. We can see from the equation that the extrema of x are reached when $y = 0$, and that the fixed points of the system are still at $(0, 0)$ (focus/node) and $(1, 0)$ (saddle)* (Exercise: prove this). *The index results show that no periodic orbit can enclose both fixed points, so the maximum value of x on a periodic orbit is 1. Thus if $a > 1$ there are no periodic orbits.*

If the flow is *nearly Hamiltonian*, and the value of H is such that the Hamiltonian orbit is closed,we can derive an approximate relation for the rate of change of H. From equation (3) we have exactly $\frac{dH}{dt} = g_2\dot{x} - g_1\dot{y}$. If g_1, g_2 are very small, then H scarcely changes and the trajectories are almost closed. If we average over many periods of the Hamiltonian flow, then the rapid oscillations in the r.h.s. are averaged out and we find an equation for the slow variation of H:

$$\frac{dH}{dt} \approx \mathcal{F}(H) = \langle g_2\dot{x} - g_1\dot{y}\rangle$$

where the brackets denote an average over a period of the Hamiltonian flow, with the quantities to be averaged evaluated for the Hamiltonian flow itself. Looking for fixed points of this reduced system is equivalent to finding periodic orbits of the nearly-Hamiltonian flow. The error in assuming that the $g_i = 0$ when calculating $\dot{\boldsymbol{x}}$ as a function of $\boldsymbol{x}$ (and H) leads to errors of order $|g_i|^2$. Alternatively one can integrate (3) around a single period of the Hamiltonian flow, giving a map $H_{n+1} = H_n + \Delta(H)$, where $\Delta(H) \approx \int_0^P g_2\dot{x} - g_1\dot{y}dt$, where to leading order the integral is calculated over a period and path of the Hamiltonian flow. Fixed points of the map give the value of H whose value approximates to a periodic orbit of the perturbed system. This method is known as the *energy balance* or *Melnikov method* though the latter is usually applied to non-autonomous perturbations to Hamiltonian flows.

Example 26 *Consider the same example as above but now with* $\epsilon \ll 1$. *Then for the Hamiltonian flow we have* $y = \pm\sqrt{2H - x^2 + \frac{2}{3}x^3}$. *The equation for the slow variation of* H *is then*

$$\frac{dH}{dt} = \frac{\epsilon}{P(H)}\oint (a-x)y^2 dt = \frac{2\epsilon\int_{x_{\min}}^{x_{\max}}(a-x)(2H - x^2 + \frac{2}{3}x^3)^{\frac{1}{2}}dx}{2\int_{x_{\min}}^{x_{\max}}(2H - x^2 + \frac{2}{3}x^3)^{-\frac{1}{2}}dx},$$

(Alternately we have $\Delta H = 2\epsilon\int_{x_{\min}}^{x_{\max}}(a-x)(2H - x^2 + \frac{2}{3}x^3)^{\frac{1}{2}}dx$*,) where* $x_{\min}$, $x_{\max}$ *are the extrema of* x *on the orbit. We can write the integrals in terms of* x *by noting that* $ydt = dx$. *Setting the l.h.s.=*0 *gives the relationship between* H *and* a. *The integrals cannot be done exactly, except in the special case where* H *takes the value* $1/6$, *for which the orbit is very close to the homoclinic orbit of the Hamiltonian flow. Then the integrals can be done exactly and we find that the critical value of* a *is* $1/7$. *Thus we see that the bound* $a = 1$ *is not very good, at least at small* ϵ*!*

4.5 Stability of Periodic Orbits

4.5.1 Floquet multipliers and Lyapunov exponents

While individual points on a periodic orbit are not stable, we can look at the stability of the whole orbit as an invariant set. We can develop a theory (**Floquet theory**) for determining whether an orbit is asymptotically stable to infinitesimal disturbances. Consider again $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$, and suppose there is a periodic orbit $\boldsymbol{x} = \hat{\boldsymbol{x}}(t)$. Letting $\boldsymbol{x} = \hat{\boldsymbol{x}} + \boldsymbol{\xi}(t)$, and linearizing, we get

$$\dot{\boldsymbol{\xi}} = \mathsf{A}(t)\boldsymbol{\xi} \ , \quad \text{where } \mathsf{A} = D\boldsymbol{f}_{\boldsymbol{x}=\hat{\boldsymbol{x}}(t)} \ . \tag{4}$$

This is a linear equation with periodic coefficients, and there is much theory concerning it. We want to know what happens to an initial disturbance $\boldsymbol{\xi}(0)$ after one period P of the original orbit. Integrating equation (4) from $t = 0$ to $t = P$, we find that due to linearity we will have the relation $\boldsymbol{\xi}(P) = \mathsf{F}\boldsymbol{\xi}(0)$, where F is a matrix that depends only on the dynamical system and on $\hat{\boldsymbol{x}}$, not on $\boldsymbol{\xi}(0)$. The eigenvalues of this matrix are called **Floquet multipliers**. One of them is always unity for an autonomous system because equation (4) is always solved by $\boldsymbol{\xi}(t) = \dot{\hat{\boldsymbol{x}}}(t)$.

Another way to find the exponents is via a map. We construct a local transversal subspace Σ through $\hat{\boldsymbol{x}}(0)$. Then all trajectories close enough to the periodic orbit intersect Σ in the same direction as the periodic orbit. Successive intersections of trajectories with Σ define a map (the **Poincaré [Return] Map** $\Phi : \Sigma \to \Sigma$. If $\boldsymbol{z}_0$ is the intersection of $\hat{\boldsymbol{x}}$ with Σ then $\Phi(\boldsymbol{z}_0) = \boldsymbol{z}_0$ so that $\boldsymbol{z}_0$ is a fixed point. Linearizing about this point we have $\Phi(\boldsymbol{z}) = \boldsymbol{z}_0 + (D\Phi)(\boldsymbol{z} - \boldsymbol{z}_0)$, where $D\Phi$ is an $(n-1) \times (n-1)$ matrix. Then the Floquet multipliers can be defined as the eigenvalues of $D\Phi$. This method suppresses the trivial unit eigenvalue described above. It is easy to prove that the choice of intersection with the periodic orbit does not affect the eigenvalues of $D\Phi$ *[Exercise]*.

We can define the stability of the orbit to small perturbations in terms of the multipliers μ_i, $i = 1, 2, \ldots, (n-1)$.

Definition 22 *(Hyperbolicity). A periodic orbit is **hyperbolic** if none of the μ_i lie on the unit circle.*

Then we have the theorem on stability analogous to that for fixed points:

Theorem 13
*(i) A periodic orbit $\hat{\boldsymbol{x}}(t)$ is asymptotically stable (a **sink**) if all the μ_i satisfy $|\mu_i| < 1$.*
(ii) If at least one of the μ_i has modulus greater than unity then the orbit is unstable (i.e. not Lyapunov stable).

Proof: very similar to that for fixed points: simple exercise.

Definition 23 *(Lyapunov exponents (Floquet exponents)). The **Lyapunov Exponents** λ_i of a periodic orbit are defined as $\lambda_i = P^{-1} \log |\mu_i|$, where the μ_i are the Floquet multipliers and P is the period. These are a measure of the rate of separation of nearby orbits.*

In $\mathbb{R}^2$ there is only one non-trivial μ, which must be real and positive (see proof of Poincaré-Bendixson Theorem).

In this case we have $\mu = \exp\left(\int_0^P \nabla \cdot \boldsymbol{f}(\hat{\boldsymbol{x}}(t))\, dt\right)$.

Proof: consider infinitesimal rectangle of length δs and width $\delta\xi$ at $\hat{\boldsymbol{x}}(0)$. Then $A(0) = \delta\xi\delta s$. By standard result for conservation of particles, $\dot{A} = \int_{\partial A} \boldsymbol{f} \cdot \boldsymbol{n} dS \sim \nabla \cdot \boldsymbol{f} \times A$. Thus $\log(A(P)/A(0)) = \mu = \exp\left(\int_0^P \nabla \cdot \boldsymbol{f}(\hat{\boldsymbol{x}}(t)\, dt\right)$, as required.

Remark: when we are in $\mathbb{R}^n$, $n \geq 3$ the μ_i may be complex; this leads to a wide variety of possible ways in which periodic orbits can lose stability:

- μ passes through $+1$. This is similar to bifurcations of fixed points (saddle-node, pitchfork etc.)

- μ passes through $\mathrm{e}^{\pm 2\mathrm{i}k\pi/n}$ $(k, n$ coprime). This leads to a new orbit that has a period approximately n times the original. IN particular when $n = 2k$ we have twice the period.

- μ passes through $\mathrm{e}^{\pm 2\mathrm{i}\nu\pi}$, ν irrational. the new solution is a 2-torus.

Example 27 *(Damped Pendulum with Torque) Here we have $\nabla \cdot \boldsymbol{f} = -k$, and so $\mu < 1$ and the periodic orbit that we have already shown to exist is thus stable.*

4.6 Example – the Van der Pol oscillator

This much studied equation can be derived from elementary electric circuit theory, incorporating a nonlinear resistance. It takes the form

$$\ddot{X} + (X^2 - \beta)\dot{X} + X = 0$$

If $\beta > 0$ then we have negative damping for small X, positive damping for large X. Thus we would imagine that oscillations grow to finite amplitude and the stabilize. It's convenient to scale the equation by writing $X = \sqrt{\beta}x$, so that

$$\ddot{x} + \beta(x^2 - 1)\dot{x} + x = 0 \ . \tag{5}$$

This equation is a special case of the **Liénard equation** $\ddot{x} + f(x)\dot{x} + g(x) = 0$. For analysis it is convenient to use the **Liénard Transformation**. We write $y = \dot{x} + F(x)$; $F(x) = \int_0^x f(s)ds$. Then in terms of x, y we have

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} y - F(x) \\ -g(x) \end{pmatrix} \tag{6}$$

For the Van der Pol system, we have $g(x) = x$, $F(x) = \beta(\frac{1}{3}x^3 - x)$ If $\beta > 0$ then it is easily seen that the (only) fixed point, at the origin, is unstable. It is hard to apply Poincaré-Bendixson, however, since the distance from the origin does not decrease monotonically at large distances. However, we expect that there is a stable limit cycle. Can use qualitative methods to show this, but first we look at the cases of small and large β.

(a) $\boldsymbol{\beta \ll 1}$. In this case the system is nearly Hamiltonian, with $H = \frac{1}{2}(x^2+y^2)$, and $g_1 = -F(x)$, $g_2 = 0$. Hence the Hamiltonian flow with Hamiltonian H is $x = \sqrt{2H}\sin t$, $y = \sqrt{2H}\cos t$, and the period P is 2π. Thus the map relation yields

$$\Delta H = -\int_0^{2\pi} \dot{y}F(x)dt = \beta\int_0^{2\pi}(x^2 - \frac{x^4}{3})dt = \beta\int_0^{2\pi}(2H\sin^2 t - \frac{4H^2}{3}\sin^4 t)dt = 2\pi\beta(H - \frac{H^2}{2})$$

Clearly $\Delta H > 0\,(< 0)$ if $H < 2\,(H > 2)$ and so the map has a stable fixed point at $H = 2$, corresponding to a periodic orbit with $x \approx 2\sin t$.

(b) $\boldsymbol{\beta \gg 1}$. If x^2 is not close to unity then the damping term is very large, so we might expect $\dot{x} = \mathcal{O}(\beta^{-1})$; in fact if we write $Y = y\beta^{-1}$ we get

$$\begin{pmatrix} \dot{x} \\ \dot{Y} \end{pmatrix} = \begin{pmatrix} \beta(Y - \frac{1}{3}x^3 + x) \\ -x\beta^{-1} \end{pmatrix}$$

so $\dot{Y}$ is very small and so Y only varies slowly. Either $\dot{x} \gg 1$ or else $Y \approx \frac{1}{3}x^3 - x$.

So the trajectory follows this curve (the *slow manifold*) until it runs out, and then moves quickly (with $\dot{x} \gg \dot{Y}$) to another branch of the curve. Oscillations of this kind are called **relaxation oscillations**. Assuming that there is such an orbit (prove below) we can find an approximation to the period. Almost all the time is taken on the slow manifold, so

$$P = 2\int_A^B dt = 2\int_{-\frac{2}{3}}^{\frac{2}{3}} \frac{dY}{\dot{Y}} = 2\int_{-\frac{2}{3}}^{\frac{2}{3}} -\frac{\beta}{x}\,dY = 2\beta\int_{-2}^{-1} \frac{(1-x^2)}{x}dx = \beta(3 - 2\ln 2)$$

In fact we can prove that there is a unique periodic orbit for all positive β. $F(x)$ has three zeroes at $-\sqrt{3}$, 0, $\sqrt{3}$ and $F(x) \to \pm\infty$ as $x \to \pm\infty$. The same methods can be used for a variety of different forms of g, F (see various old Tripos questions). If we define $G(x) = \int_0^x g(s)ds = \frac{1}{2}x^2$ and $H = \frac{1}{2}(y^2 + x^2)$ then $\dot{H} = -\beta x^2(1 - x^2/3)$. Consider the nullclines $\dot{x} = 0$ ($y = F(x)$) and $\dot{y} = 0$ ($x = 0$) . We show that trajectories enter the four regions of the plane bounded by the nullclines, and that we can define a return map for H which has at least one stable fixed point.

First note that if $x > 0$, $y > F(x)$ x is increasing, y is decreasing. Thus trajectory must cross the curve $y = F(x)$. Now both x and y are decreasing, and $\dot{y}$ decreases in magnitude as the origin is approached. Thus trajectory must cross into $x < 0$ where now $\dot{y} > 0$. Repeating the argument we see that trajectory must return to $x > 0$, $y > F(x)$ eventually. For sufficiently small orbits around the origin $\dot{H} > 0$ and so we see that for sufficiently small orbits H increases, in fact by looking at the time reversed problem with $\beta \to -\beta$ we can show that eventually $x^2 + y^2 > 3$ and remains so.

Need to show that large enough initial conditions lead to H decreasing. Consider diagram. Then

$$H(B) - H(A) = \int_A^B \frac{\dot{H}}{\dot{x}}\, dx = \int_{-\sqrt{3}}^{\sqrt{3}} \frac{\beta x^2(1 - x^2/3)}{y - \beta x(x^2/3 - 1)} dx$$

This is positive but a monotonically decreasing function of y. On the other hand

$$H(C) - H(B) = \int_B^C \frac{\dot{H}}{\dot{y}}\, dy = \int_B^C -\beta x(1 - \frac{x^2}{3})dy$$

This is negative and gets more negative as $y(\sqrt{3})$ increases. So $H(C) - H(A)$ decreases monotonically with $y(A)$ and so there is a value of $y(A)$ for which $y(C) = y(A)$. By symmetry this leads to a unique closed orbit for some $y(A)$.

5 Bifurcations

5.1 Introduction

We return now to the notion of dynamical systems depending on one or more **parameters** $\mu_1, \mu_2, \ldots$. We are interested in parameter values for which the system is not **structurally stable**. Recall the definitions:

Definition 24 *(Topological equivalence). Two vector fields* $\boldsymbol{f}$ $\boldsymbol{g}$ *and associated flows* $\phi^{\boldsymbol{f}}$, $\phi^{\boldsymbol{g}}$ *are* ***topologically equivalent*** *if* $\exists$ *a homeomorphism (1-1, continuous, with continuous inverse)* $h : \mathbb{R}^n \to \mathbb{R}^n$, *and a map* $\tau(t, \boldsymbol{x}) \to \mathbb{R}$, *strictly increasing on* t, *s.t.*

$$\tau(t+s, \boldsymbol{x}) = \tau(s, \boldsymbol{x}) + \tau(t, \phi_s^{\boldsymbol{f}}(\boldsymbol{x}))\ , \text{ and } \phi^{\boldsymbol{g}}_{\tau(t,\boldsymbol{x})} h(\boldsymbol{x}) = h(\phi_t^{\boldsymbol{f}}(\boldsymbol{x}))$$

(Structural stability). The vector field $\boldsymbol{f}$ *is* ***structurally stable*** *if for all twice differentiable vector fields* $\boldsymbol{v}$ $\exists \epsilon_v > 0$ *such that* $\boldsymbol{f}$ *is topologically equivalent to* $\boldsymbol{f} + \epsilon \boldsymbol{v}$ *for all* $0 < \epsilon < \epsilon_v$.

It turns out that if we change parameters for a given $\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu})$ then we will have structural stability in general except on surfaces where $\boldsymbol{f}$ is not structurally stable. We define a **bifurcation point** as a point in parameter space where $\boldsymbol{f}$ is not structurally stable. A **bifurcation** (change in structure of the solution) will occur when the parameters are varied to pass through these points. If we plot e.g the amplitude of the fixed points and periodic orbits etc as the parameters are varied, as functions of the parameters, this is called a **bifurcation diagram**.

5.2 Stationary bifurcations in $\mathbb{R}^2$

5.2.1 One-dimensional bifurcations

Stationary bifurcations occur when one eigenvalue of the Jacobian for a given fixed point is zero. These are best understood initially in one dimension. Suppose we have an o.d.e. $\dot{x} = f(x, \boldsymbol{\mu})$, and that when $\boldsymbol{\mu} = 0$ the equation has a fixed point at the origin, which is non-hyperbolic. Thus we have $f(0, \mathbf{0}) = \partial f_x(0, \mathbf{0}) = 0$ (subscripts denote partial derivatives). There are three possible types of bifurcation involving one parameter. The first is the generic case, and the others will be found when there are particular restrictions on the system.

1. **Saddle-Node Bifurcation.** $\dot{x} = \mu - x^2$. We have $x = +\sqrt{\mu}$ (stable) and $x = -\sqrt{\mu}$ (unstable) when $\mu > 0$, a saddle-node at 0 when $\mu = 0$ and no fixed points for $\mu < 0$.

2. **Transcritical Bifurcation.** $\dot{x} = \mu x - x^2$. There are fixed points at $x = 0, \mu$ which exchange stability at $\mu = 0$.

3. **Pitchfork Bifurcation.** $\dot{x} = \mu x \mp x^3$. Fixed point at $x = 0$, also at $x = \pm\sqrt{\pm\mu}$ when $\pm\mu > 0$. The $(\ldots - x^3)$ case is called *supercritical*, the other case *subcritical*; in the supercritical(subcritical) case the bifurcating solutions are both stable(unstable).

More insight is gained by considering two-parameter families. Supppose we have

$$\dot{x} = \mu_1 + \mu_2 x - x^2 \; .$$

This includes both families (1) and (2) as special cases. Fixed points are at

$$x = \frac{\mu_2 \pm \sqrt{\mu_2^2 + 4\mu_1}}{2} \text{ provided that } \mu_2^2 + 4\mu_1 > 0$$

There ia a single non-hyperbolic fixed point on the parabola $\mu_2^2 + 4\mu_1 = 0$, and none to the left of this curve.

Clearly passing *through* any point of the parabola yields a saddle-node bifurcation. To see a transcritical bifurcation, it is necessary for the path in parameter space to be *tangential* to the parabola. e.g. If we vary μ_2 at fixed μ_1 then in general only bifurcations are saddle-nodes.

We say that the saddle-node bifurcation is **codimension 1** (i.e. bifurcation set has a dimension one less than that of the parameter space). The o.d.e. $\dot{x} = \mu - x^2$ is a **universal unfolding** of the saddle node $\dot{x} = -x^2$. We can also say that $\dot{x} = \mu - x^2$ is the **normal form** for the saddle-node bifurcation, in the sense that generic vector fields near the saddle-node can be reduced to this form by a near-identity diffeomorphism.

Example 28 *(Reduction to standard form.) Consider a more general two-parameter family*

$$\dot{x} = \mu_1 + \mu_2 x - x^2$$

and try a change of variable $y = x - \alpha$, so that

$$\dot{y} = (\mu_1 + \mu_2\alpha - \alpha^2) + y(\mu_2 - 2\alpha) - y^2$$

Choosing $\alpha = \mu_2/2$ gives $\dot{y} = (\mu_1 + \mu_2^2/4) - y^2$, which is in the standard form. The three-parameter system

$$\dot{x} = \mu_1 + \mu_2 x - \mu_3 x^2$$

can be reduced scaling time so that $T = \mu_3$ to

$$\frac{dx}{dT} = \frac{\mu_1}{\mu_3} + \frac{\mu_2}{\mu_3}x - x^2$$

and so to the standard form.

We can now treat the case of general $f(x,\mu)$ provided that (as we assume) f can be expanded in a Taylor series in both x and μ near the non-hyperbolic (bifurcation) point $(0,0)$. At this point we already have $f(0,0) = 0 = f_x(0,0)$. Now suppose, what is generally true, that $f_{xx} \neq 0$, $f_\mu \neq 0$ at this point. Now we expand f in a double Taylor series about $(0,0)$:

$$\begin{aligned} f(x,\mu) &= f(0,0) + xf_x(0,0) + \mu f_\mu(0,0) \\ &\quad + \frac{x^2}{2}f_{xx}(0,0) + x\mu f_{x\mu}(0,0) + \frac{\mu^2}{2}f_{\mu\mu}(0,0) + \mathcal{O}(3) \ . \end{aligned}$$

Rearranging, we have

$$f(x,\mu) = (\mu f_\mu + \mathcal{O}(\mu^2)) + x(\mu f_{x\mu}) + \frac{x^2}{2}f_{xx} + \mathcal{O}(3)$$

and this is in the correct form to be reduced to the standard saddle-node equation.

There are important special cases, non-generic in the space of all problems, that nonetheless are of physical importance.

- **Transcritical bifurcation**. If the system is such that $f(0,\mu) = 0$ for all μ (or can be put into this form by a change of variable), then $f_\mu(0,0) = 0$, and we have instead of the above

  $$f(x,\mu) = \mu_2 x + \mu_3 x^2 + \mathcal{O}(3) \ ,$$

  with all the higher order terms vanishing when $x = 0$. This is in the standard form for a transcritical bifurcation.

- **Pitchfork bifurcation**. If the system has a *symmetry*; that is if the equation is unchanged under an operation on the space variables whose square is the identity, then simple bifurcations are pitchforks. In $\mathbb{R}$ the only such operation is $x \to -x$; for the equations to be invariant f must be odd in x. Then expanding in the same way we get $f(x,\mu) = \mu_2 x + \mu_4 x^3 + \mathcal{O}(x^5)$. (In higher dimensions symmetries can take more complicated form. For example, the system $\dot{x} = \mu x - xy\,,\ \dot{y} = -y + x^2$ has symmetry under $x \to -x$, $y \to y$).

The saddle-node bifurcation is robust under small changes of parameters as shown above. But transcritical and pitchfork bifurcations depend on the vanishing of terms, and therefore change under small perturbations.

$$\dot{x} = \epsilon + \mu x - x^2$$

$$\dot{x} = \epsilon + \mu x - x^3$$

$$\dot{x} = \mu x + \epsilon x^2 - x^3$$

The most general 'unfolding' of the pitchfork bifurcation takes two parameters:

$$\dot{x} = \epsilon_1 + \mu x + \epsilon_2 x^2 - x^3$$

(Diagram: exercise)

In $\mathbb{R}^2$, The fixed point is non-hyperbolic if there is at least one purely imaginary (or zero) eigenvalue.

Consider now the situation where $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu)$ has a non-hyperbolic fixed point at the origin, for some value μ_0 of μ. There are four cases:

- (i) $\lambda_1 = 0,\ \Re\lambda_j \neq 0,\ j \neq 1$. This is a **simple**, or **steady-state** bifurcation, and is essentially the same as the 1D examples shown above. WE show this below when we discuss the centre manifold.
- (ii) $\lambda_{1,2} = \pm i\omega,\ \Re\lambda_j \neq 0,\ j \neq 1, 2$. This is an **oscillatory** or **Hopf** bifurcation, and leads to the growth of oscillations.
- (iii) and (iv) There are two zero eigenvalues. Canonical form of matrix A is either $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (**double-zero bifurcation**), or $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ (**Takens-Bogdanov bifurcation**). They have quite different properties and are not seen generically as they need two parameters to be varied.

Note that there are extra technical requirements on the way the eigenvalues change with μ (e.g. for (i) we must have $d\lambda_1/d\mu \neq 0$ at $\mu = \mu_0$). We shall only look at (i) and (ii) in detail.

5.3 The Centre Manifold

Consider first the simple bifurcation. By analogy with the hyperbolic case, when $\mu = \mu_0$ the linear system has a subspace on which the solutions decay (or grow) exponentially, and another in which the dynamics is non-hyperbolic (the **centre eigenspace** as defined below). E.g. for a saddle node we have $\dot{x} = x^2$, $\dot{y} = -y$. By analogy with the stable and unstable manifolds and their relation to the stable and unstable subspaces in the hyperbolic case, we might expect a manifold to exist (the **centre manifold**), which is tangent to the centre eigenspace at the origin, and on which the dynamics correspond to that in the centre eigenspace.

Example 29 *Consider the non-hyperbolic system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} x^2 + xy + y^2 \\ -y + x^2 + xy \end{pmatrix} .$$

*The linear system has $\dot{y} = -y$ and so trajectories approach $y = 0$, which is the centre eigenspace. In this space $\dot{x} = x^2$. Thus hope there is an invariant manifold tangent to $y = 0$ at the origin on which $\dot{x} \sim x^2$, (the **Centre Manifold** or **CM**) (see handout)*

It turns out that this is true (see Centre Manifold Theorem below), though CM's are not like stable and unstable manifolds in that they are not unique (see example sheet 2).

We can find the CM in this example, assuming existence, by expansion. Suppose it is of form $y = p(x) \equiv a_2x^2 + a_3x^3 + a_4x^4 + \ldots$. Then $\dot{y} = -p + x^2 + xp = \dot{x}p_x = (x^2 + xp + p^2)p_x$. This o.d.e. for $p(x)$ cannot be solved in general, but substitute the expansion for p, equate coefficients and get $a_2 = 1$, $a_3 = -1$, $a_4 = 0$. Thus the CM is given by $p(x) = x^2 - x^3 + \mathcal{O}(5)$. The dynamics on the CM is then given by replacing y by p in the equation for $\dot{x}$:

$$\dot{x} = x^2 + xp + p^2 = x^2 + x^3 - x^4 + \mathcal{O}(6) + x^4 - 2x^5 + \mathcal{O}(6) = x^2 + x^3 - 2x^5 + \mathcal{O}(6) \ ,$$

and so close to the origin we have $\dot{x} \sim x^2$ as expected so that we have a saddle-node for the nonlinear system.

The existence of a centre manifold is guaranteed under appropriate conditions by the **Centre Manifold Theorem**:

Theorem 14 *(Centre Manifold Theorem). Given a dynamical system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x})$ in $\mathbb{R}^n$ with a non-hyperbolic fixed point at the origin, let E^c be the (generalised) linear eigenspace corresponding to eigenvalues of $\mathsf{A} = D\boldsymbol{f}|_0$ with zero real part (the **centre subspace**), and E^h the* *complement of E^c (the* ***hyperbolic subspace****). Choose a coordinate system $(\boldsymbol{c}, \boldsymbol{h})$, $\boldsymbol{c} \in E^c$, $\boldsymbol{h} \in E^h$ and write the o.d.e. in the form*

$$\begin{pmatrix} \dot{\boldsymbol{c}} \\ \dot{\boldsymbol{h}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{C}(\boldsymbol{c}, \boldsymbol{h}) \\ \boldsymbol{H}(\boldsymbol{c}, \boldsymbol{h}) \end{pmatrix} .$$

Then $\exists$ a function $\boldsymbol{p} : E^c \to E^h$ with graph $\boldsymbol{h} = \boldsymbol{p}(\boldsymbol{c})$, called the ***centre manifold*** *which has properties:*

- *(i) is tangent to E^c at 0;*
- *(ii) is locally invariant under $\boldsymbol{f}$;*
- *(iii) dynamics is topologically equivalent to*

$$\begin{pmatrix} \dot{\boldsymbol{c}} \\ \dot{\boldsymbol{h}} \end{pmatrix} = \begin{pmatrix} \boldsymbol{C}(\boldsymbol{c}, \boldsymbol{p}(\boldsymbol{c})) \\ \left.\dfrac{\partial \boldsymbol{H}}{\partial \boldsymbol{h}}\right|_0 \boldsymbol{h} \end{pmatrix}$$

- *(iv) $\boldsymbol{p}(\boldsymbol{c})$ can be approximated by a polynomial in $\boldsymbol{c}$ in some neighbourhood of* 0.

Thus in Example 29, $c = x$, $h = y$ and $p(x) = x^2 - x^3 + \ldots$; local dynamics is equivalent to $\dot{x} = x^2 + xp(x) + p(x)^2$, $\dot{y} = -y$

This is very helpful for non-hyperbolic systems, but we believe that such reductions ought to be possible *near*, and not just *at*, bifurcation points. We can use the centre manifold theorem by means of a trick:

To our original system $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \boldsymbol{\mu})$, which now has a *hyperbolic* fixed point at 0, adjoin the equations $\dot{\boldsymbol{\mu}} = 0$. We now have a system in $\mathbb{R}^{(n+m)}$, where m is the number of parameters. In this new system the terms giving the linearized growth rates as functions $\boldsymbol{\mu}$ will be *nonlinear*, and so we are at a non-hyperbolic fixed point of the extended system, and can use the CM theorem to reduce the system.

Example 30 *Consider the system*

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} \mu + x^2 + xy + y^2 \\ 2\mu - y + x^2 + xy \end{pmatrix} .$$

Regarding μ as a variable we have

$$\begin{pmatrix} \dot{x} \\ \dot{\mu} \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 2 & -1 \end{pmatrix} \begin{pmatrix} x \\ \mu \\ y \end{pmatrix} \quad + \text{nonlinear terms.}$$

Thus the centre eigenspace is $y = 2\mu$, which is a plane in $\mathbb{R}^3$, and the CM is of the form

$$y = p(x, \mu) = 2\mu + a_{20}x^2 + a_{11}x\mu + a_{02}\mu^2 + \dots \ ,$$

we have

$$\dot{y} = 2\mu - p + x^2 + xp = -\frac{\partial p}{\partial x}(\mu + x^2 + xy + y^2) + \frac{\partial p}{\partial \mu} \cdot 0 \ , \ \text{or}$$

$$\begin{aligned} 2\mu + (2\mu + a_{20}x^2 + a_{11}x\mu + a_{02}\mu^2 + \dots)(x-1) + x^2 \\ = \big(\mu + x^2 + x(2\mu + a_{20}x^2 + a_{11}x\mu + a_{02}\mu^2 + \dots) + \\ + (2\mu + a_{20}x^2 + a_{11}x\mu + a_{02}\mu^2 + \dots)^2\big)\,(2a_{20}x + a_1\mu + \dots) \end{aligned} .$$

Equating coefficients, we find $[x^2]: \ a_{20} = 1$, $[x\mu]: \ a_{11} = 0$, $[\mu^2]: \ a_{02} = 0$. Thus the CM is given by $y = p(x, \mu) = 2\mu + x^2 + O(3)$, and the dynamics on the extended CM is given by

$$\begin{aligned} \dot{x} &= \mu + x^2 + x(2\mu + \dots) + 4\mu^2 + \dots \\ \dot{\mu} &= 0 \end{aligned}$$

This is clearly (cf. the one-parameter families above) a saddle-node bifurcation when $\mu = 0$. Of course none of this is really necessary if only the local behaviour is desired, but it gives one confidence to proceed.

Thus we have the important result that for a dynamical system in $\mathbb{R}^n$, with a simple bifurcation at μ_0 (i.e. the system has a single zero eigenvalue at $\mu = \mu_0$, and (wlog) all eigenvalues are in the l.h. half plane for $\mu = \mu_0^-$, and there is just one eigenvalue in the r.h. half-plane for $\mu = \mu_0^+$), there is a centre manifold, the dynamics is essentially one-dimensional and the bifurcation is generically a saddle-node.

5.4 Oscillatory/Hopf Bifurcations in $\mathbb{R}^2$

The simplest model of a Hopf (oscillatory) bifurcation is given by the system (in polar coordinates) $\dot{r} = \mu r - r^3$, $\dot{\theta} = 1$. For $\mu < 0$ we have a stable focus, and for $\mu > 0$ an unstable focus and a stable periodic orbit $r = \sqrt{\mu}$. This is a **supercritical Hopf**. If instead we have $\dot{r} = \mu r + r^3$ then the periodic orbit exists when $\mu < 0$, and is unstable (**subcritical Hopf**).

It turns out that generically all dynamical systems can be put into this form in the neighbourhood of a Hopf bifurcation. We need a technical definition of this bifurcation.

Definition 25 *(Hopf bifurcation). Suppose we have a dynamical system* $\dot{\boldsymbol{x}} = \boldsymbol{f}(\boldsymbol{x}, \mu) = (f(\boldsymbol{x}, \mu), g(\boldsymbol{x}, \mu))$ *which at* $\mu = 0$ *has a fixed point at* 0, *and has linearization* A *satisfying* $\det\mathsf{A} > 0$, $\mathrm{Tr}\mathsf{A} = 0$ *[so that the linearization has eigenvalues* $\lambda_{1,2}(0) = \pm i\omega$*], and that* $d(\Re(\lambda_{1,2})/d\mu > 0$ *at* $\mu = 0$. *Then provided a constant* γ, *defined as the value at* $\mu = 0$ *of*

$$\frac{1}{16}\left(f_{xxx} + g_{yyy} + f_{xyy} + g_{xxy} + \omega^{-1}(f_{xy}(f_{xx} + f_{yy}) - g_{xy}(g_{xx} + g_{yy}) - f_{xx}g_{xx} + f_{yy}g_{yy})\right)$$

is not equal to zero, then there is a Hopf bifurcation at $\mu = 0$ *and there is a stable limit cycle for* $\mu = 0^+$ *if* $\gamma < 0$ ***(supercritical Hopf bifurcation)*** *and an unstable limit cycle for* $\mu = 0^-$ *if* $\gamma > 0$ ***(subcritical Hopf bifurcation)***.

Remark: because A is not singular at $\mu = 0$, there is no change in the number of fixed points for small $|\mu|$. Most questions will not demand a knowledge of the exact form of γ, but will usually be in recognizable form. Rather than derive the formula (see Glendinning if interested) we show how the equation can be brought into standard, i.e. normal form at $\mu = 0$ by a near identity diffeomorphism.

Suppose then that $\mu = 0$ and choose canonical coordinates so that $\mathsf{A} = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$. Then writing (in these coordinates) $z = x + iy$, the linear part of the problem is $\dot{z} = i\omega z$. Then the full equation must take the form

$$\dot{z} = i\omega z + \alpha_1 z^2 + \alpha_2 z z^* + \alpha_3 z^{*2} + \mathcal{O}(3) \ .$$

Define a new complex variable ξ by $\xi = z + a_1 z^2 + a_2 z z^* + a_3 z^{*2}$. We try to choose $a_{1,2,3}$ so that $\dot{\xi} = i\omega\xi + \mathcal{O}(3)$. In fact

$$\begin{aligned} \dot{\xi} &= \dot{z}(1 + 2a_1 z + a_2 z^*) + \dot{z^*}(a_2 z + 2a_3 z^*) \text{ so correct to } \mathcal{O}(3), \\ i\omega(z + a_1 z^2 + a_2 z z^* + a_3 z^{*2}) &= (i\omega z + \alpha_1 z^2 + \alpha_2 z z^* + \alpha_3 z^{*2})(1 + 2a_1 z + a_2 z^*) \\ &\quad + (-i\omega z^* + \alpha_1^* z^{*2} + \alpha_2^* z z^* + \alpha_3^* z^2)(a_2 z + 2a_3 z^*) \end{aligned}$$

Equating coefficients of the quadratic terms we get

$$i\omega a_1 = \alpha_1 + 2i\omega a_1; \ i\omega a_2 = \alpha_2 + i\omega a_2 - i\omega a_2; \ i\omega a_3 = \alpha_3 - 2i\omega a_3$$

and clearly these equations can be solved. Thus in the transformed system there are no quadratic terms. Attempting to perform the same trick at cubic order, we find that all the cubic terms can be removed except the term $\propto z^2 z^*$ [Exercise]. Thus after all the transformations have been completed we are left with the equation

$$\dot{z} = i\omega z + \nu z^2 z^* + h.o.t., \text{ or } \dot{r} = \Re(\nu) r^3, \ \dot{\theta} = \omega + \Im(\nu) r^2$$

and $\gamma \propto \Re(\nu)$.

To find the canonical equation when $\mu \neq 0$ we can either use the same ideas on the extended CM that we used for the simple bifurcation, or just add the relevant linear terms; which can be shown to yield the same result in non-degenerate cases.

Note that the normal form is appropriate only when r is sufficiently small. Because the constant part of $\dot{\theta}$ is of order unity there will be no fixed points in a neighbourhood of the origin.

Example 31 *Find the nature of the Hopf bifurcation for the system* $\dot z = (\mu + i\omega)z + \alpha z^2 + \beta |z|^2$.

Clearly bifurcation point is at $\mu = 0$, *so choose this value and, guided by above analysis, choose* $\xi = z + az^2 + b|z|^2$. *Then*

$$\dot\xi = (i\omega z + \alpha z^2 + \beta|z|^2)(1 + 2az + bz^*) + (-i\omega z^* + \alpha^* z^{*2} + \beta^*|z|^2)bz$$

$$= i\omega z + (\alpha + 2ia)z^2 + \beta|z|^2 + |z|^2 z(2a\beta + \alpha b + b\beta^*) + \text{ other cubic}$$

$$= i\omega(\xi - az^2 - b|z|^2) + (\alpha + 2ia)z^2 + \beta|z|^2 + |z|^2 z(2a\beta + \alpha b + b\beta^*) + \text{ other cubic}$$

Now choose $i\omega a = -\alpha :\ i\omega b = \beta$. Then quadratic terms vanish, and

$$= i\omega\xi + (i\omega)^{-1}(|\beta|^2 - \alpha\beta)|\xi|^2\xi + \text{other cubic} + \mathcal{O}(4)$$

so that $\Re(\nu) = -\Im(\alpha\beta)/\omega$.

In fact it can be shown by successive transformations that the canonical form for the dynamics on the CM for a Hopf bifurcation is $\dot z = zF(|z|^2)$, where F is a complex valued function with $F(0) = i\omega$. This allows the treatment of degenerate situations for which $\Re(\nu) = 0$.

For higher dimensional systems the CM theorem can be invoked to show that in the neighbourhood of a Hopf bifurcation there is a 2-dimensional (extended) CM on which the dynamics can be put into the above form.

5.5 *Bifurcations of periodic orbits*

(a) Homoclinic bifurcation

This is the simplest 'global' bifurcation and occurs when the stable and unstable manifolds of a saddle point intersect at a critical value μ_0 of the parameter. There are two 'flavours' possible:

(b) 'Andronov bifurcation'

A very similar type of bifurcation arises when a saddle-node develops on a periodic orbit (**"Andronov bifurcation"**).]

In each case, periodic orbits are produced as a result if the bifurcation, and we would like to know the stability of these orbits.

6 Bifurcations in Maps

6.1 Examples of maps

We have seen that the study of the dynamics of flows near periodic orbits can be naturally expressed in terms of a map (the Poincaré map). This helps to motivate the study of maps in their own right. There are other motivations too.

- **Maps of the interval.** These originally arose as discrete versions of 1D flows, but have much richer structure. Suppose we have the ode $\dot{x} = f$; if we have discrete time intervals $t_0, t_1, \ldots, t_n \equiv t_0 + n\Delta t, \ldots$, with $x(t_n) = x_n$ then Euler's method gives $x_{n+1} = x_n + \Delta t f(x_n)$. Alternatively can think of x_n as a population at the nth generation. Generalising gives the general nonlinear map $x_{n+1} = g(x_n)$. Famous example is the **Logistic (quadratic) map** :

$$x_{n+1} = \mu x_n(1 - x_n) \quad 0 < x_n \leq 1 : \quad 1 \leq \mu \leq 4.$$

Another important map (since calculations are easy!) is the **Tent (piecewise linear) map**. An example is

$$x_{n+1} = \left\{ \begin{array}{ll} \mu x_n & 0 \leq x_n \leq \frac{1}{2} \\ \mu(1 - x_n) & \frac{1}{2} \leq x_n \leq 1 \end{array} \right\} \quad 0 < x_n \leq 1 : \quad 1 \leq \mu \leq 2$$

- **Circle Maps.** These are motivated by Poincaré maps for the instability of periodic orbits. Use units in which the circumference of the circle is unity.

  Rotation. $x_{n+1} = x_n + \omega$ [mod 1]. This is *periodic* if ω rational, *aperiodic* if ω irrational.

  Standard Map. $x_{n+1} = x_n + \mu \sin 2\pi x_n$ [mod 1]. When μ small this is almost the identity, for larger μ get more interesting behaviour.

  Sawtooth Map (Bernoulli Shift). $x_{n+1} = 2x_n$ [mod 1]. We can find the solution for any x_0 by expressing x_n in binary form: $x_n = 0.i_1 i_2 i_3 \ldots i_n \ldots$ where the i_j are zero or unity. Then $x_{n+1} = 0.i_2 i_3 \ldots$. So if $x_0 = 0$, $x_n = 0$, if x_0 is rational then binary expansion repeats and so $x_n = x_0$ for some n (periodic) while if x_0 is irrational solution is aperiodic. This map is a prototype of chaotic behaviour.

  The shift map can be seen as a special case of the logistic map. Put $x_n = \sin^2 \pi\theta_n$, with θ_n satisfying the shift map. Then $x_{n+1} = \sin^2 2\pi\theta_n = 4x_n(1 - x_n)$ so that x_n satisfies the logistic map with $n = 4$. In fact the these maps are **topogically conjugate**:

  Definition 26 *(Topological conjugacy for maps.) Two maps F and G are* ***topologically conjugate*** *if there exists a smooth invertible map H such that $F = H^{-1} \cdot G \cdot H$.*

- **Maps of the Plane.** These come from Poincaré maps of flows in $\mathbb{R}^3$. As such they should be invertible, or at least have unique inverse when an inverse exists.

  Hénon Map

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} 1 + y_n - a x_n^2 \\ b x_n \end{pmatrix}$$

For appropriate choice of a, b this has 'strange' behaviour with fractal structure.

Baker Map. Composed of two maps:

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{cases} \begin{pmatrix} 2x_n \\ \frac{1}{2} y_n \end{pmatrix} & 0 < x_n < \frac{1}{2},\ 0 < y_n < 1 \\[2ex] \begin{pmatrix} 2 - 2x_n \\ 1 - \frac{1}{2} y_n \end{pmatrix} & \frac{1}{2} < x_n < 1,\ 0 < y_n < 1 \end{cases}$$

So this is a map of $[0 < x < 1,\ 0 < y < 1]$ into itself.

Horseshoe Map (Smale). Best seen in terms of diagram:

Set of points that do not leave box form a fractal Cantor-type set.

6.2 Fixed points, cycles and stability

We can define in a way analogous to flows **fixed points** and **periodic points (cycles)** of a map:

Definition 27 *Given a map* $\boldsymbol{x}_{n+1} = \boldsymbol{f}(\boldsymbol{x}_n)$, *a* ***fixed point*** $\boldsymbol{x}_0$ *satisfies* $\boldsymbol{x}_0 = \boldsymbol{f}(\boldsymbol{x}_0)$, *while a* ***periodic point*** *with* ***period*** n *if* $\boldsymbol{x}_0 = \boldsymbol{f}^n(\boldsymbol{x}_0)$, $\boldsymbol{x}_0 \neq \boldsymbol{f}^m(\boldsymbol{x}_0)$, $m < n$. *Here* $\boldsymbol{f}^2 \equiv \boldsymbol{f} \circ \boldsymbol{f}$ *(composition),* $\boldsymbol{f}^n = \boldsymbol{f} \circ \boldsymbol{f}^{n-1}$. *A set of periodic points* $\{\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_n = \boldsymbol{x}_1\}$ *is called* ***cycle****.*

We can define ω-points (and α-points for invertible maps with inverses) as points of accumulation of iterates of $\boldsymbol{x}_n$ as $n \to \infty$ ($-\infty$ for α-points).

We can define notions of Lyapunov stability and quasi-asymptotic stability just as for flows. The notions can be combined into the notion of an **attractor**.

Definition 28 *Suppose* $\mathcal{A}$ *is a closed set mapped into itself by* $\boldsymbol{f}$. *This could be a fixed point, cycle or a more exotic set. We suppose that* $\boldsymbol{f}$ *is continuous at points of* $\mathcal{A}$. *Then* $\mathcal{A}$ *is an* ***attractor*** *if*

(i) For any nbd $\mathcal{U}$ *of* $\mathcal{A}$ $\exists$ *a nbd* $\mathcal{V}$ *of* $\mathcal{A}$ *such that for any* $\boldsymbol{x} \in \mathcal{V}$, $\boldsymbol{f}^n(\boldsymbol{x}) \in \mathcal{U} \quad \forall n \geq 0$ *(Lyapunov stability);*

(ii) $\exists$ *an nbd* $\mathcal{W}$ *of* $\mathcal{A}$ *such that for any* $\boldsymbol{x} \in \mathcal{W}$ *and any set* $\mathcal{A}'$ *containing* $\mathcal{A}$ $\exists n_0$ *such that* $\boldsymbol{f}^n(\boldsymbol{x}) \in \mathcal{A}' \quad \forall n > n_0$ *(quasi-asymptotic stability).*

Consider a fixed point (chosen to be at the origin) of $\boldsymbol{x}_{n+1} = \boldsymbol{f}(\boldsymbol{x}_n)$, where $\boldsymbol{f}$ has continuous first derivative at and near the origin. Then we can form the jacobian matrix $\mathsf{A} = D\boldsymbol{f}_0$. If we suppose that eigenvalues of A are distinct or there is a complete set of eigenvectors, then we can show that if all the eigenvalues λ of A satisfy $|\lambda| < 1$, then the origin is an attractor.

Proof: Let $\boldsymbol{z}_j$ be the left eigenvectors of A (possibly complex). Then let $\mathcal{V}_n = \sum v_i |\boldsymbol{z}_i \cdot \boldsymbol{x}_n|^2$, for some positive set of numbers v_i. Then $\mathcal{V}_{n+1} = \sum v_i |\boldsymbol{z}_i \cdot \mathsf{A}\boldsymbol{x}_n|^2 + \mathcal{O}(|\boldsymbol{x}_n|^3)$. The first term on the rhs is $\sum v_i |\lambda_i|^2 |\boldsymbol{z}_i \cdot \boldsymbol{x}_n|^2 < a^2 \mathcal{V}_n$, where $a^2 = \max |\lambda_i|^2 < (1-\epsilon)$, $\epsilon > 0$. We can choose $\mathcal{V}_n$ sufficiently small that the cubic remainder term is less than $\epsilon \mathcal{V}_n / 2$, say, and so $\mathcal{V}_n \to 0$, $n \to \infty$. Conversely if any eigenvalue λ has $|\lambda| > 1$ the fixed point is a **repellor** (neither QAS nor Lyapunov stable). *Proof: exercise.*

For a cycle of least period r each point of the cycle is a fixed point of the map $\boldsymbol{f}^r$, so stability is determined by the eigenvalues of the jacobian of this map. If we write for each point $\boldsymbol{x}^{(j)}$ of the cycle $\mathsf{A}^{(j)} \equiv D\boldsymbol{f}(\boldsymbol{x}^{(j)})$ the linearization of the map about the cycle can be written $\boldsymbol{\xi}_{n+1} = \mathsf{A}^{(n)}\boldsymbol{\xi}_n$ and so the linearization of $\boldsymbol{f}^r$ is just $\mathsf{A}^{(r)}\mathsf{A}^{(r-1)}\ldots\mathsf{A}^{(1)}$ (can also see this from the chain rule).

6.3 Local bifurcations in 1-dimensional maps

Now the jacobians are just real numbers, and bifurcations must occur when eigenvalues of a map passes through the unit circle, and so through ± 1. We can classify these bifurcations just as for 1-dimensional flows.

- **Saddle-Node**. If we suppose as before that $f = f(x, \mu)$ and that $f(0,0) = 0,\ f_x(0,0) = 1$ then expanding in x, μ as before get

$$x_{n+1} = x_n + \mu f_\mu + \frac{1}{2}x_n^2 f_{xx} + x_n \mu f_{x\mu} + \ldots$$

and truncating, shifting the origin and rescaling we get the canonical form

$$x_{n+1} = x_n + \mu - x_n^2$$

which has no fixed points when $\mu < 0$ and two fixed points when $\mu > 0$, for which $x = \pm\sqrt{\mu}$ and the jacobian is $1 - 2x$. So for $0 < \mu < 1$ one of the fixed points is stable and the other unstable. (something new happens for $\mu > 1$, but the normal form is supposed to be valid for sufficiently small μ).

- **Transcritical bifurcation**. Now suppose in addition that $f_\mu(0,0) = 0$; then we have

$$x_{n+1} = x_n + \frac{1}{2}(x_n^2 f_{xx} + \mu^2 f_{\mu\mu}) + x_n \mu f_{x\mu} + \dots$$

Truncating and seeking fixed points, we need $(x^2 f_{xx} + \mu^2 f_{\mu\mu} + 2x\mu f_{x\mu}) = 0$ and this is only possible if $f_{x\mu}^2 > f_{xx} f_{\mu\mu}$. Otherwise there are no fixed points and so no bifurcation. If satisfied can write truncated system in canonical form

$$x_{n+1} = x_n - (x_n - a\mu)(x_n - b\mu)$$

for some a, b. There are two lines of fixed points $x = a\mu,\ x = b\mu$ crossing at origin when $\mu = 0$, each exchanges stability as μ passes through zero. If we write $y_n = x_n - a\mu$, say get even simpler form $y_{n+1} = y_n - y_n(y_n - c\mu)$.

- **Pitchfork bifurcation**. This is achieved as before when the eigenvalue is unity, and if there is a symmetry (equivariance) under $x \to -x$, in which case only odd terms occur in the expansion of f. Then we have

$$x_{n+1} = x_n + \mu x_n \pm x_n^3 + \mathcal{O}(5)$$

We also get a pitchfork when $f_\mu = f_{xx} = 0$, $f_{\mu x}, f_{xxx} \neq 0$, in which case the correction is $\mathcal{O}(4)$.

- **Period-doubling bifurcation**. The remaining case has eigenvalue -1 at the bifurcation point. Thus in the general case

$$x_{n+1} = -x_n + a' + b'x_n + c'x_n^2 + d'x_n^3 + \mathcal{O}(4)$$

where a', b', c', d' are functions of μ, with $a', b' = \mathcal{O}(\mu)$, $c' = \frac{1}{2}f_{xx} + \mathcal{O}(\mu)$, $d' = \frac{1}{6}f_{xxx} + \mathcal{O}(\mu)$. There is a fixed point of this system at $x = x^*$, where $2x^* \approx a' + b'x^* + c'x^{*2} + d'x^{*3}$. Changing variable so that $y = x - x^*$, we get

$$y_{n+1} = -y_n + by_n + cy_n^2 + dy_n^3$$

where $b = b' + a'c' + \mathcal{O}(\mu^2)$ and $c, d = c', d' + \mathcal{O}(\mu)$. now f.p. is unstable when $b < 0$, stable if $b > 0$. Consider the map f^2. Then $y_{n+2} = -y_{n+1}(1-b) + cy_{n+1}^2 + dy_{n+1}^3$, so that

$$y_{n+2} = (1-b)((1-b)y_n - cy_n^2 - dy_n^3) + c(y_n^2(1-b)^2 - 2c(1-b)y_n^3) - d(1-b)^3y_n^3 + \ldots$$

simplifying and keeping only leading terms in b we get

$$y_{n+2} = (1-2b)y_n - bcy_n^2 - 2(c^2+d)y_n^3$$

If we make a change of variable of the form $z_n = y_n + \alpha b$, then correct to order b we can remove the quadratic term to give $z_{n+2} = (1-2b)z_n - 2(c^2+d)z_n^3$. Thus there is a pitchfork bifurcation of f^2 at $b = 0$. The nonzero fixed points of f^2 correspond to a 2-cycle of f. It can be shown that these cycles are stable if the origin is unstable and *vice verca*.

7 Chaos

7.1 Introduction

What do we mean by chaos? Two main concepts (a) in a chaotic system, initially nearby orbits separate, and also we have some sort of mixing of even the smallest sets by iterating the map.

Consider a continuous map $f : I \to I$ of a bounded interval $I \subset \mathbb{R}$ into itself, and with $\Lambda \subset I$ a set invariant under f:

Definition 29 *Sensitive dependence on initial conditions. f has* ***sensitive dependence on initial conditions****[SDIC] on Λ if $\exists\, \delta > 0$ s.t. for any $x \in \Lambda$ and $\epsilon > 0$ $\exists y \in \Lambda$ and $n > 0$ s.t. $|y - x| < \epsilon$ $|f^n(x) - f^n(y)| > \delta$*

Note that not all points in the neighbourhood have to separate in this way, and nothing said about exponential divergence.

Definition 30 *Topological Transitivity. The map f above is* ***topologically transitive****[TT] on Λ if for any pair of open sets K_1, K_2 s.t. $K_i \cap \Lambda \neq \emptyset$, $i = 1, 2$, $\exists\, n > 0$ s.t. $f^n(K_1) \cap K_2 \neq \emptyset$. This means that there are orbits that are dense in Λ, i.e. come arbitrarily close to every point of Λ, and so Λ cannot be decomposed into disjoint invariant sets*

Example 32 *(TT but not SDIC).The rotation map $x_{n+1} = x_n + \omega[\text{mod } 1]$ is TT on $[0, 1]$ if ω is irrational (though not SDIC).*

Example 33 *(SDIC but not TT). The map $x_{n+1} = 2x_n$ $(|x_n| < \frac{1}{2})$, $x_{n+1} = 2(\text{sign}(x_n) - x_n)$ $(\frac{1}{2} < |x_n| < 1)$ has SDIC on $[-1, 1]$ since $|f'| = 2$, but is not TT on $[-1, 1]$ as $x = 0$ is invariant.*

There are two apparently quite different definitions of chaos, though they are more similar than they look.

Definition 31 *(Chaos [Devaney]). $f : I \to I$ is* ***chaotic*** *on Λ if (i) f is SDIC on Λ; (ii) f is TT on Λ; (iii) periodic points of f are dense in Λ.*

The second definition depends on the *Horseshoe property.*

Definition 32 *(Horseshoe property). $f : I \to I$ has a* ***horseshoe*** *if $\exists\, J \subseteq I$ and disjoint open subintervals K_1, K_2 of J s.t. $f(K_i) = J$ for $i = 1, 2$.*

If f has a horseshoe it can be shown that (i) f^n has at least 2^n periodic points; (ii) f has periodic points of every period; (iii) f has an uncountable number of aperiodic orbits.

Definition 33 *(Chaos[Glendinning]) A continous map $f : I \to I$ is* ***chaotic*** *if f^n has a horseshoe for some $n \geq 1$.*

This last definition allows for maps with stable fixed points, but demands exponential divergence of nearby trajectories. It will be shown that Chaos[G]⇒Chaos[D].

7.2 The Sawtooth Map (Bernoulli shift)

As an example consider the **sawtooth map** $f(x) = 2x[\text{mod } 1]$.

(a) f clearly has a horseshoe with $K_1 = (0, \frac{1}{2})$, $K_2 = (\frac{1}{2}, 1)$. (Note, *open* sets). So f is chaotic[G].

(b) (i) As before use binary expansion so if $x_n = 0.a_1a_2a_3\ldots$ then $x_{n+1} = 0.a_2a_3\ldots$. So if expansion repeats after n terms we have a cycle of period n. There are 2^n of these. Clearly also periodic points are dense in $[0, 1]$.

If $x \neq y$ then suppose x and y differ first in the $(n+1)$th place; then for $r \leq n$ $|f^r(x) - f^r(y)| = 2^r|x - y|[\text{mod}\,1]$.

Choose $\delta < \frac{1}{2}$. Then given any $x \in [0, 1)$ and $\epsilon > 0$ choose n so that $2^{-n-1} < \epsilon$ and choose y to differ from x *only* in the $n + 1$ binary place. Then $|f^n(y) - f^n(x)| = \frac{1}{2} > \delta$, so f is SDIC.

(ii) Choose any point $x = 0.a_1a_2\dots$; then for another point $z = 0.b_1b_2\dots$ choose $y_N = 0.b_1b_2\dots b_Na_1a_2\dots$. Then $f^N(y_N) = x$ and we can make y_N arbitrarily close to z, by taking N sufficiently large, so map is TT.

(iii) Now define $x_N = 0.a_1a_2\dots a_Na_1a_2\dots a_Na_1\dots$ (or $0.\dot{a}_1a_2\dots\dot{a}_N$. This is certainly a periodic point and we can make x_N arbitrarily close to x. So periodic points are dense in $[0, 1)$.

Thus f is chaotic[D].

In fact f is a very effective mixer. Define $f(\frac{1}{2}) = 1$ and suppose x is *not* a preimage of 1/2, i.e. that there is no n s.t. $f^n(x) = \frac{1}{2}$. Thus x does not end with a infinite sequence of 0's (or 1's). Choose m s.t. $a_{m+1} = 0$ Then it is easy to see that $y = 0.a_1\dots a_m\,0\,0\dots < x < z = 0.a_1\dots a_m\,1\,0,\dots$, and that $f^n(y) = 0$, $f^n(z) = \frac{1}{2}$. Thus $f^{n+1}((y, z)) = (0, 1)$, and so any arbitrarily small neighbourhood of x can be mapped into the whole interval. (this is because the map is 1-1 on each half range and by construction y, z lie in the same half range and so do $f(y), f(z)$.)

7.3 Horseshoes, symbolic dynamics and the shift map

Aim to show that a map with the horseshoe property acts on a certain invariant set Λ in the same way that as the shift map.

Suppose a continous map f has a horseshoe on an interval $J \subset \mathbb{R}$ and define the closed intervals $I = \overline{J}$, $I_i = \overline{K_i}$, $i = 1, 2$.

Assume that f is monotonic on I_i, $I_1 \cap I_2 = \emptyset$ and that $f(x) \in I \Rightarrow x \in I_1$ or I_2. Define $\Lambda = \{x : f^n(x) \in I, \forall n \geq 0\}$. Clearly $x \in \Lambda \Rightarrow f(x) \in \Lambda$, and $x \in \Lambda \Rightarrow x = f(y)$ for some $y \in I$ (Intermediate value theorem). Thus $f(\Lambda) = \Lambda$ so Λ is invariant.

For each $x \in \Lambda$ $f^n(x) \in I \Rightarrow f^{n-1}(x) \in I_1$ or I_2. Define $a_n = 0$ if $f^{n-1}(x) \in I_1$, $a_n = 1$ if $f^{n-1} \in I_2$. Thus x corresponds to the sequence $a_1a_2\dots$ while $f(x)$ corresponds to $a_2a_3\dots$. This is essentially the same as the action of the shift map on binary expansions of numbers in $[0, 1]$. There are small differences as $0.111\dots$ and $1.000\dots$ are the same number but different symbol sequences; however this does not affect the proofs of TT and SDIC. Thus chaos[G]⇒horseshoe⇒chaos[D].

What does Λ look like? The set $f^{-1}(I) \equiv \{x \in I : \ f(x) \in I\}$ comprises two disjoint closed intervals. So $f^{-n}(I)$ has 2^n closed intervals and $\Lambda = \bigcap_{n=1}^{\infty} f^{-n}(I)$. Limit gives a closed set with an uncountable number of points but length zero, cf. the middle third Cantor set.

7.4 Period 3 implies chaos

Recall the Intermediate Value Theorem: If $f : [a,b] \to \diagdown$ is continuous, and $f(a) = c$, $f(b) = d$, then $\forall y \in [c,d] \exists x \in [a,b]$ s.t. $f(x) = y$. In particular if $f(x) - x$ changes sign on $[a,b]$ then $\exists x_0 \in [a,b]$ then $\exists x_0 \in [a,b]$ s.t. $f(x_0) = x_0$. We can now prove the remarkable theorem:

Theorem 15 *(Period 3 implies chaos). If a continuous map f on $I \subseteq \mathbb{R}$ has a 3-cycle then f^2 has a horseshoe and so f is chaotic.*

Proof: let $x_1 < x_2 < x_3$ be the elements of the 3-cycle. wlog suppose $f(x_1) = x_2, f(x_2) = x_3, f(x_3) = x_1$ (otherwise consider instead $-f(-x)$).

$$f(x_2) = x_3 > x_2, \ f(x_3) = x_1 < x_2 \ \Rightarrow \ \exists z \in (x_2, x_3) \text{ s.t. } f(z) = z$$

$$f(x_1) = x_2 < z, \ f(x_2) = x_3 > z \ \Rightarrow \ \exists y \in (x_1, x_2) \text{ s.t. } f(y) = z$$

Thus $f^2(y) = f^2(z) = z > y$ and $f^2(x_2) = x_1 < y$, so $\exists$ a smallest $r \in (y, x_2)$ s.t. $f^2(r) = y$, and $\exists$ largest $s \in (x_2, z)$ s.t. $f^2(s) = y$. Thus f^2 has a horseshoe with $K_1 = (y, r)$, $K_2 = (s, z)$ and $J = (y, z)$.

7.5 Existence of N-cycles

Have shown that f^2 has a horseshoe if there is a 3-cycle, which implies existence of cycles for f^2 of all periods. In fact can show that f has cycles of all periods.

Lemma. Recall that if f s continuous and $V \subseteq f(U)$, where U, V are closed intervals, the $\exists$ a closed interval $K \subseteq U$ s.t. $F(K) = V$.

Theorem 16 *If a continuous map f on $\subseteq \mathbb{R}$ has a 3-cycle the it has an N-cycle $\forall\, N \geq 1$.*

Take setup as before.

$N = 1$: $f(x_3), x_2 < x_3 = f(x_2) \Rightarrow$ there is a fixed point of the map.

$N > 1$: let $I_L = [x_1, x_2]$, $I_R = [x_2, x_3]$. Then $f(I_L) \supseteq I_R$, $f(I_R) \supseteq I_L \cup I_R$.

Choose $J_N = I_R$. Then define $F(J_{N-1}) = J_N$, $J_{N-1} \subseteq I_L$ by the Lemma. Similarly define $J_{N-2}, \ldots, J_0 \subseteq I_R$ by $f(J_i) = J_{i+1}$.

$f^N(J_0) = I_R \Rightarrow \exists a, b \in J_0$ s.t. $f^N(a) = x_2,\ f^N(b) = x_3$. But $J_0 \subseteq I_R$ so $a \geq f^N(a),\ b \leq f^N(b)$ Thus by IVT there is a fixed point $z \in [a, b]$ of f^N. $f^{N-1}(z) \in I_L$. so only possibility is if $f^{N-1}(z) = x_2 \Rightarrow f^N(z) = x_3 = z$.

But $f^2(x_3) = x_2 \neq x_3$. Also $f(x_3) = x_1 \notin I_R$ so violating construction. Thus $f^{N-1}(z) \neq x_2$ so must not be in I_R This shows that one iterate is definitely different from all others so an N-cycle.

The statements $f(I_l) \supseteq I_R$, $f(I_R) \supseteq I_L \cup I_R$ can be shown as a *directed graph*:

Cycles exist when there are closed paths in the diagram.

Example 34 *Suppose there is a 4-cycle $x_1 = f(x_4) < x_3 = f(x_2) < x_2 = f(x_1) < x_4 = f(x_3)$. Let $I_A = [x_1, x_3]$, $I_B = [x_3, x_2]$, $I_C = [x_2, x_4]$. Only fixed points in I_C and 2-cycles between I_A, I_B possible.*

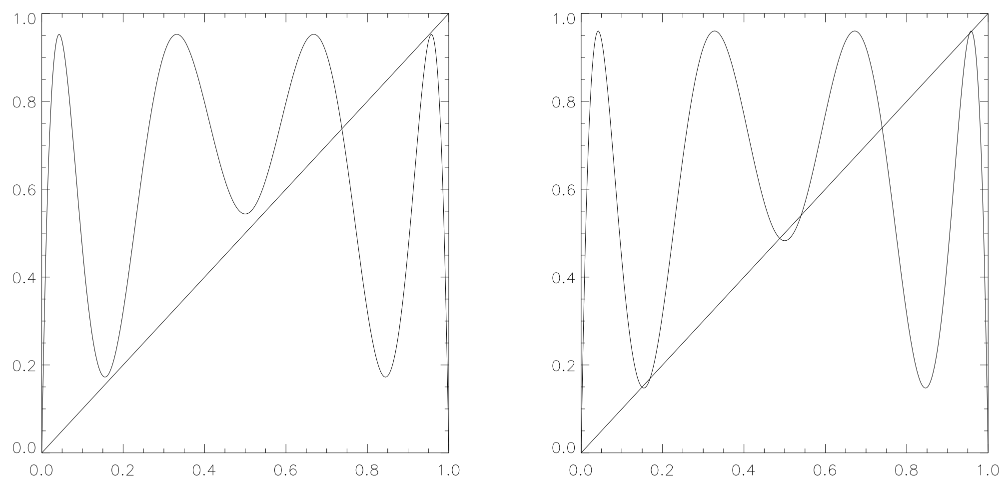
A remarkable result due to Sharkovsky can be proved (proof not in course) by similar methods to the above.

Theorem 17 *(Sharkovsky.) If $f : \mathbb{R} \to \mathbb{R}$ is continous, f has a k-cycle and $l \lhd\ k$ in the following ordering, then f also has an l-cycle:*

$$
\begin{array}{l}
1 \lhd\ 2 \lhd\ 2^2 \lhd\ 2^3 \lhd\ 2^4 \lhd\ \ldots \\
\ldots \\
\ldots \lhd\ 2^3 \cdot 9 \lhd\ 2^3 \cdot 7 \lhd\ 2^3 \cdot 5 \lhd\ 2^3 \cdot 3 \\
\ldots \lhd\ 2^2 \cdot 9 \lhd\ 2^2 \cdot 7 \lhd\ 2^2 \cdot 5 \lhd\ 2^2 \cdot 3 \\
\ldots \lhd\ 2 \cdot 9 \lhd\ 2 \cdot 7 \lhd\ 2 \cdot 5 \lhd\ 2 \cdot 3 \\
\ldots \lhd\ 9 \lhd\ 7 \lhd\ 5 \lhd\ 3
\end{array}
$$

This has many implications, not least that if f has a cycle of period 3 then it has cycles of *all* periods, as proved separately above.

Note that the theorem says nothing about the stability of the cycles. However for the logistic equation at least we know that all cycles either arise from a period-doubling bifurcation, or in the case of the odd-period cycles as a saddle-node bifurcation so they are all stable in some range!!



The period-three orbit for the logistic map. Shown is the map f^3 where $f(x) = \mu x(1-x)$. Top picture: $\mu = 3.81$, bottom picture; $\mu = 3.84$.

7.6 The Tent Map

A more interesting map, because it depends on a parameter, is the **tent map** $f(x) = \mu[\frac{1}{2} - |x - \frac{1}{2}|]$

Fixed point at 0 stable for $\mu < 1$. Choose $\mu \in (1, 2]$. In this range the origin is unstable and the interval $[0, 1]$ is mapped into itself. There is a f.p. $x_0 = \mu/(1+\mu)$, which is always unstable (exercise).

We show that the map is chaotic[G] when $1 < \mu \leq 2$.

Step 1. All non-zero orbits eventually enter and stay in the interval $A = [f^2(\frac{1}{2}), f(\frac{1}{2})] = [\mu(1-\mu/2, \mu/2]$. Note that if the *preimage* of x_0, $[x_{-1} = 1/(1+\mu)] \in A$; i.e. if $1/(1+\mu) > \mu(1-\mu/2)$ then $\mu > \sqrt{2}$.

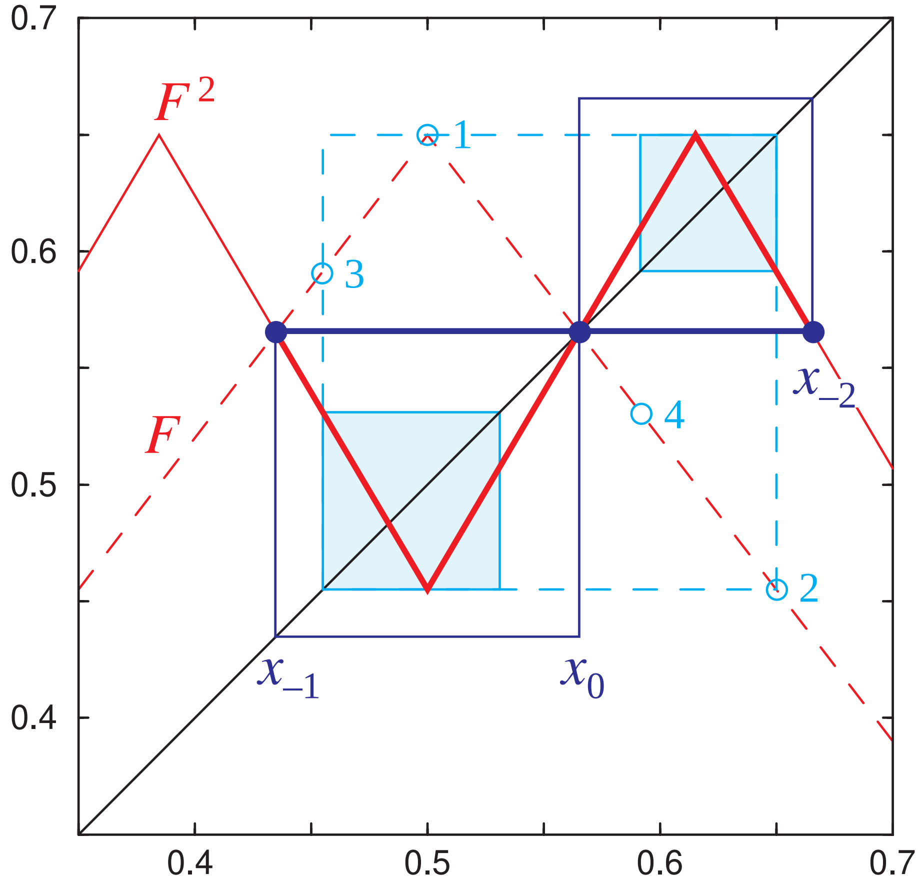
Step 2. Now consider $f^2(x)$:

$$
\begin{aligned}
f^2(x) &= \mu^2 && 0 \leq x \leq \frac{1}{2}\mu \\
&\mu(1-\mu x) && \frac{1}{2}\mu \leq x \leq \frac{1}{2} \\
&\mu(1-\mu(1-x)) && \frac{1}{2} \leq x \leq 1 - \frac{1}{2}\mu \\
&\mu^2(1-x) && 1 - \frac{1}{2}\mu \leq x \leq 1
\end{aligned}
$$

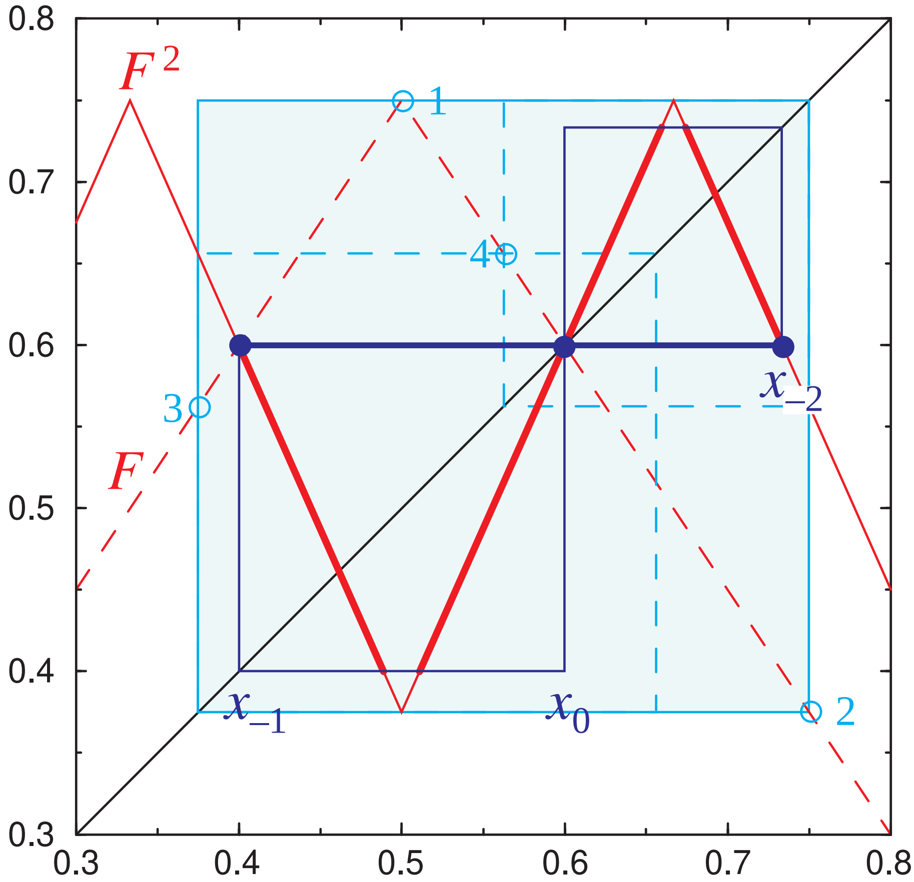
Let x_{-2} be the preimage under f of x_{-1} (Or the preimage under f^2 of the fixed point x_0) in $x > \frac{1}{2}$. Then $x_{-2} = (\mu^2 + \mu - 1)/\mu(\mu + 1)$.

Step 3. By a change of coordinates we can see that f^2 acts like a tent map with parameter μ^2 on the two intervals $J_L = [x_{-1}, x_0]$ and $J_R = [x_0, x_{-2}]$.

Two different cases:



For $\mu < \sqrt{2}$, (see picture for $\mu = 1.3$), f^2 gives tent maps with parameter μ^2 on the intervals $[x_{-1}, x_0]$ and $[x_0, x_{-2}]$, and the attracting set (shaded) has two components defined by $f^i(\frac{1}{2})$, $i = 1, \ldots, 4$ (circles).



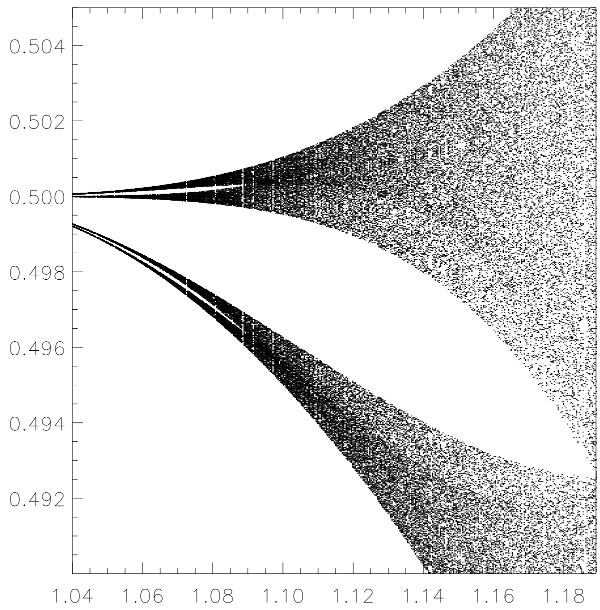
For $\mu > \sqrt{2}$, (see picture for $\mu = 1.5$), f^2 has horseshoes on the intervals $[x_{-1}, x_0]$ and $[x_0, x_{-2}]$, and the attracting set (shaded) has one component since $f^4(\frac{1}{2}) > f^3(\frac{1}{2})$.

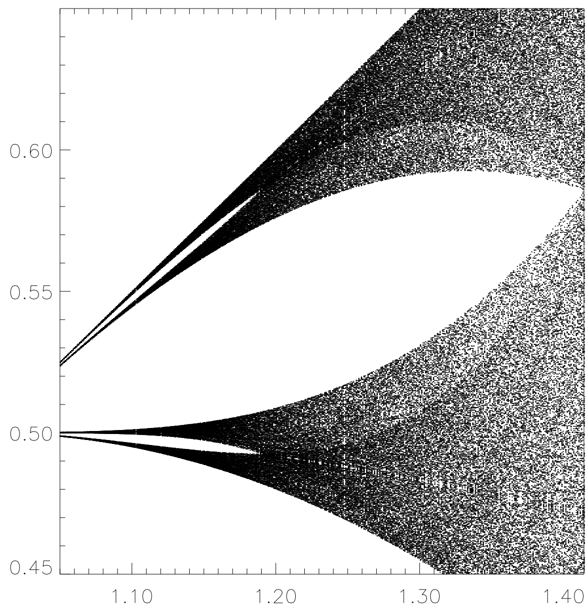
Step 4. Now suppose that $\sqrt{2} \leq \mu^n < 2$. Now $f^{2^{n+1}}$ has a horseshoe on 2^n intervals that are permuted by f. Proof: apply above arguments inductively.

The union of all these intervals is the complete range $1 < \mu \leq 2$.

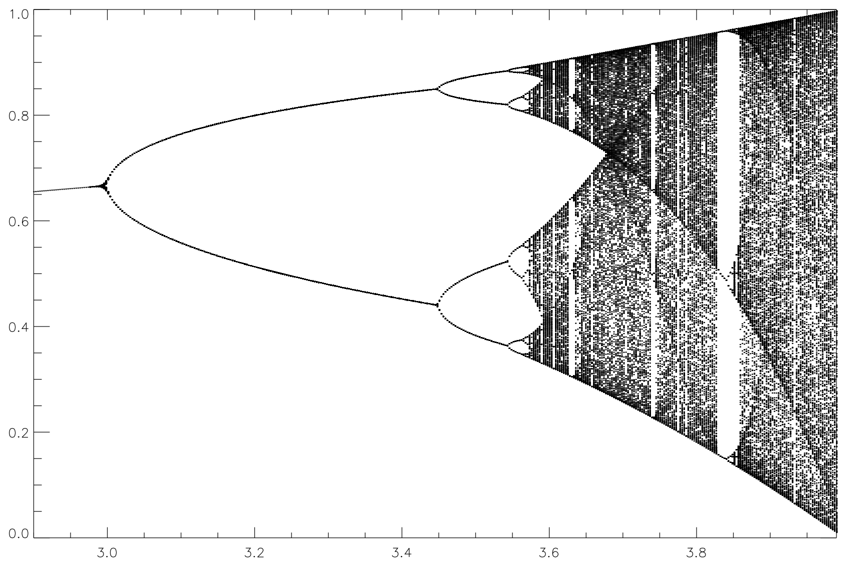
The following images of the (μ, x) plane were produced by interating the tent map for $O(1000)$ iterations to allow the orbit to settle towards the chaotic attractor, and then plotting the next $O(500)$ points. (The vertical stripiness is an annoying artefact of the printer resolution!)

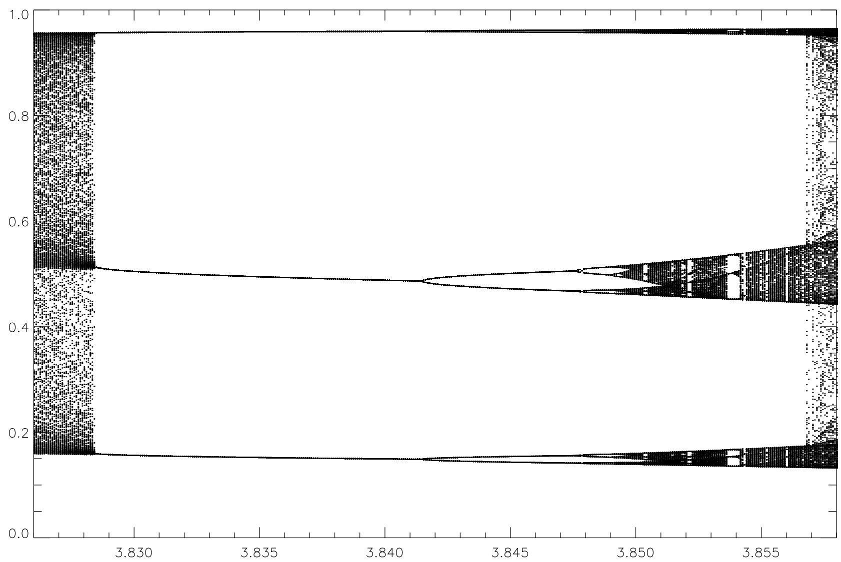
The attracting set contains 2^n intervals in $2^{1/2^{n+1}} \leq \mu < 2^{1/2^n}$.

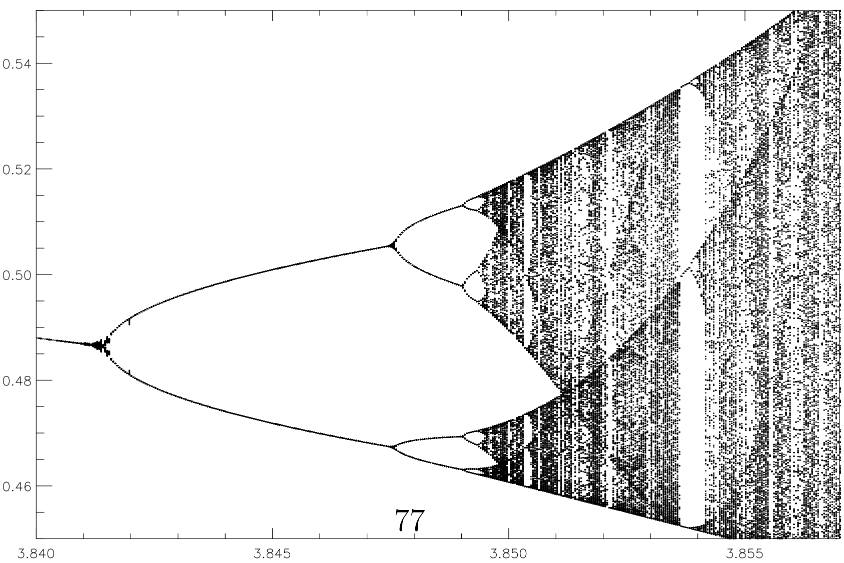




7.7 The Logistic Map







Here $x_{n+1} = \mu x_n(1-x_n)$, $0 < \mu \leq 4$. There is one non-trivial fixed point at $\bar{x} = (\mu-1)/\mu$. jacobian is $\mu(1-2\bar{x}) = 2-\mu$, so there is a period-doubling bifurcation at $\mu = 3$. By looking at iterate of map we find a further bifurcation (to period 4) at $\mu = 1+\sqrt{6}$; to period 8 at $\mu \approx 3.544$. Call μ_k point of bifurcation to cycle of period 2^k; then it was shown by Feigenbaum (1978) that

$$\delta = \lim_{k\to\infty}\left(\frac{\mu_k-\mu_{k-1}}{\mu_{k+1}-\mu_k}\right) = 4.6692\ldots \text{ Feigenbaum's constant;} \quad \mu_k \to \mu_\infty = 3.5699\ldots$$

This ratio turns out to be a *universal* constant for all one-humped (unimodal) maps with a quadratic maximum (taken at $x = \frac{1}{2}$). First we give some general results about such maps.

7.7.1 Unimodal Maps

Definition 34 *A **unimodal map** on the interval* $[a,b]$ *is a continuous map* $F : [a,b]$ *into* $[a,b]$ *such that (i)* $F(a) = F(b) = a$ *and (ii)* $\exists c \in (a,b)$ *such that* F *is strictly increasing on* $[a,c]$ *and strictly decreasing on* $[c,b]$*. i.e.*

Note: a map of the form

is effectively unimodal under $x \mapsto -x$ and $F \mapsto -F$.

Definition 35 *An **orientation reversing fixed point** (ORFP) of a unimodal map* F *is a fixed point in the interval* (c,b) *where* F *is decreasing.*

Lemmas

(1) If $F(c) \leq c$ then all solutions tend to fixed points, which lie in $[a, F(c)]$.

(2) If $F(c) > c$ then there is a unique ORFP $x_0 \in (c, F(c))$.

(3) If $F(c) > c$ then orbits either tend to fixed points in $[a, F^2(c)]$ or are attracted into $[F^2(c), F(c)]$.

Proof

(1) $F([a, c]) = F([c, b]) = [a, F(c)] \subseteq [a, c]$. So after one iteration $x_1 \in [a, c]$, where $x < y \Longleftrightarrow F(x) < F(y)$.

If $x_1 < F(x_1)$ then x_i increases monotonically to the nearest fixed point.

If $x_1 > F(x_1)$ then x_i decreases monotonically to the nearest fixed point.

(2) Apply the IVT to $F(x) - x$ on $[c, F(c)]$ noting that $F(c) > c \Rightarrow F^2(c) < F(c)$.

(3) Exercise. (Cases split on whether $F^3(c) < F^2(c)$ or vice versa.)

Theorem 18 *If F has an ORFP x_0 then $\exists\, x_{-1} \in (a, c)$ and $x_{-2} \in (c, b)$ such that $F(x_{-2}) = x_{-1}$ and $F(x_{-1}) = x_0$. Moreover,*

- *either (i) F^2 has a horseshoe on $J_L \equiv [x_{-1}, x_0]$ and $J_R \equiv [x_0, x_{-2}]$*
- *or (ii) all solutions tend to fixed points of F^2*
- *or (iii) F^2 is a unimodal map with an ORFP on both J_L and J_R.*

Proof

$x_0 \in (c,b) \Rightarrow F(c) > F(x_0) = x_0 > F(b) = F(a) \overset{IVT}{\Longrightarrow} \exists\, x_{-1} \in (a,c)$ such that $F(x_{-1}) = x_0$.

$x_{-1} \in (a,c) \Rightarrow F(b) = a < x_{-1} < x_0 = F(x_0) \overset{IVT}{\Longrightarrow} \exists\, x_{-2} \in (x_0, b)$ such that $F(x_{-2}) = x_{-1}$.

Thus $F^2(x_{-2}) = F^2(x_{-1}) = F^2(x_0) = x_0$. Also $x \in [x_{-1}, x_0] \Rightarrow F^2(x) \in [F^2(c), x_0]$ and $x \in [x_0, x_{-2}] \Rightarrow F^2(x) \in [x_0, F(c)]$ i.e. F^2 has the graph

(i) If $F^2(c) < x_{-1}$ (equivalent to $F(c) > x_{-2}$) then F^2 has horseshoes.

(ii) If $F^2(c) > c$ then all solutions on $J_L \cup J_R$ tend to fixed points of F^2. All solutions on $[a, x_{-1}] \cup [x_{-2}, b]$ either tend to fixed points of F or are attracted into $[F^2(c), F(c)] \subset J_L \cup J_R$.

(iii) If $x_{-1} < F^2(c) < c$ then F^2 is a unimodal map on J_L and J_R with ORFPs that give a 2-cycle of F, and the attracting set consists of two disjoint subintervals.

Applying Theorem 1 successively to $F^2, F^4, F^8, \ldots$ we deduce that

Theorem 19 *If F has an ORFP then*

- *either (i) $\exists N$ such that F^{2^N} has a horseshoe and F is chaotic*
- *or (ii) $\exists N$ such that all solutions tend to fixed points of F^{2^N} and F has 2^m-cycles for $0 \le m \le N-1$*
- *or (iii) there are (mostly unstable) 2^m-cycles $\forall m$, and the attracting set is a Cantor set formed by the infinite intersection of the attracting subintervals of F^{2^m}.*

Proof

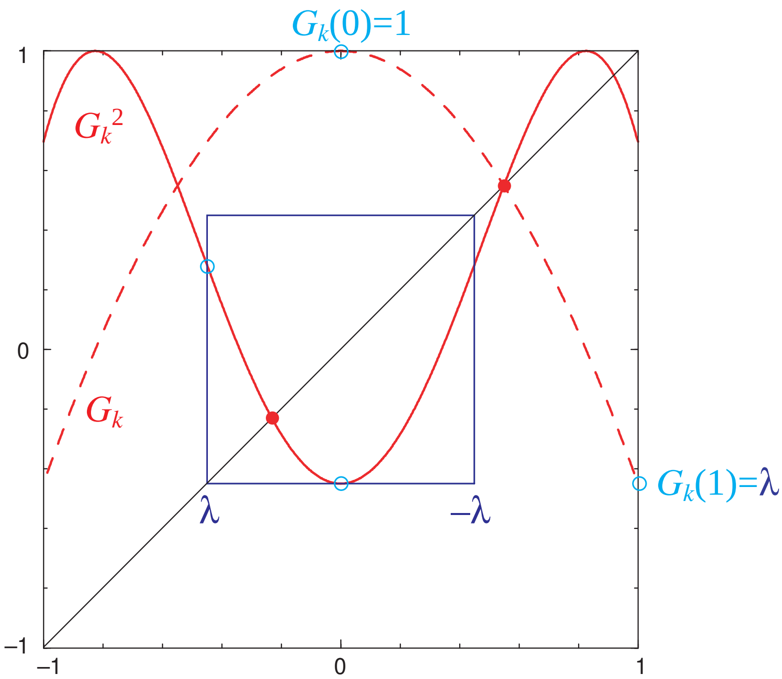
Induction (except for the comment on stability which depends on the following).

7.7.2 Scaling Invariance and Feigenbaum's Constant

If we write $G_k = F^{2^k}$ then in situation (iii) of Theorem 19 the successive subgraphs of $G_{k+1}, G_{k+2}, \ldots$ all seem to look the same after renormalisation and all seem to have the same properties. This suggests that we look for a graph that is invariant under iteration and renormalisation:

Suppose w.l.o.g that $c = 0$ and after renormalisation $G_k(0) = 1$ $\forall k$. (This is easier than scaling the sub-interval so that the end x_0 is always at 1.)

Suppose, for simplicity, that $G(x) = G(-x) = 1 + ax^2 + bx^4 + \ldots$. The graphs of G_k and G_k^2 look like



Let λ be the value of $G_k^2(0) = G_k(1)$.

Renormalise G_k^2 so that $G_{k+1}(0) = 1$ by defining

$$G_{k+1}(y) = \frac{G_k^2(\lambda y)}{\lambda} \equiv \mathcal{T}[G_k] \text{ say, where } \lambda = G_k^2(0)$$

To get an approximation to $\overline{G}$ try a truncated series expansion:

e.g. $G_k = 1 + a_k x^2 + o(x^2)$ with $G_k(1) = 1 + a_k = \lambda_k$

$$\Rightarrow \quad G_{k+1} = \mathcal{T}[G_k] = \frac{1 + a_k\{1 + a_k[(1+a_k)x]^2\}^2}{1 + a_k} = 1 + 2a_k^2(1+a_k)x^2 + o(x^2)$$

i.e. $a_{k+1} = 2a_k^2(1+a_k)$, which has an unstable fixed point $\overline{a} = -\frac{1}{2}(1+\sqrt{3}) = -1.37 \Rightarrow \overline{\lambda} = -0.37$, where the Jacobian is $4 + \sqrt{3} = 5.73$

e.g. $G_k = 1 + a_k x^2 + b_k x^4 + o(x^4)$ with $G_k(1) = 1 + a_k + b_k = \lambda_k$ gives the 2D map

$$a_{k+1} = 2a_k(a_k + 2b_k)\lambda_k \qquad b_{k+1} = (2a_k b_k + a_k^3 + 4b_k^2 + 6a_k^2 b_k)\lambda_k^3$$

which has a fixed point $\overline{a} = -1.5222$, $\overline{b} = 0.1276$, $\overline{\lambda} = -0.3946$, where the Jacobian has eigenvalues 4.844 and -0.49.

In fact, numerical solution shows that the functional map $\mathcal{T}$ has a fixed point $\overline{G} = \mathcal{T}[\overline{G}]$, where

$$\overline{G} = 1 - 1.52736x^2 + 0.10482x^4 - 0.02671x^6 + \ldots, \qquad \overline{\lambda} = \overline{G}(1) = -0.3995$$

Numerical linearisation about $\mathcal{T}[\overline{G}] = \overline{G}$ gives a single eigenvalue $\delta = 4.6692016\ldots$ outside the unit circle, and an infinite spectrum of eigenvalues inside the unit circle. Hence situation (iii) of Theorem 19 (renormalisation possible infinitely often) is unstable in one direction; the stable manifold occupies 'all but one dimension' of the possible space of functions.

The map $G_0 = \mu_\infty x(1-x)$, $\mu_\infty = 3.5700\ldots$, is on the stable manifold of $\overline{G}$, but a small perturbation grows to give situation (i) if $\mu > \mu_\infty$ (G_N has a horseshoe for some N) or situation (iii) if $\mu < \mu_\infty$ (G_N has no ORFP for some N and cycle lengths divide 2^N).

If $\mu_\infty - \mu = O(\delta^{-N})$ then it takes $O(N)$ renormalisations for the perturbation to grow to $O(1)$ and eliminate the ORFP, thus explaining why $\mu_\infty - \mu_k \sim A\delta^{-k}$ as $k \to \infty$.