Mathematical Methods I
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Example Sheet 3

1. Use the Cauchy–Schwarz inequality and the properties of the inner product to prove the triangle inequality
$$|\boldsymbol{x}+\boldsymbol{y}| \leqslant |\boldsymbol{x}| + |\boldsymbol{y}|$$
for a complex vector space, where $|\boldsymbol{x}|$ is the norm of the vector $\boldsymbol{x}$. Under what conditions does equality hold?

2. Given a set of vectors $\boldsymbol{u}_1, \boldsymbol{u}_2, \ldots, \boldsymbol{u}_m$ $(m \geqslant n)$ that span an n-dimensional vector space, show that an orthogonal basis may be constructed by the Gram–Schmidt procedure
$$\boldsymbol{e}_1 = \boldsymbol{u}_1 \, ,$$
$$\boldsymbol{e}_r = \boldsymbol{u}_r - \sum_{s=1}^{r-1} \frac{\boldsymbol{e}_s \cdot \boldsymbol{u}_r}{\boldsymbol{e}_s \cdot \boldsymbol{e}_s} \boldsymbol{e}_s \quad \text{for } r > 1 \, .$$
What is the interpretation if any of the vectors $\boldsymbol{e}_r$ vanishes?

   Find an orthonormal basis for the subspace of a four-dimensional Euclidean space spanned by the three vectors with components $(1,1,0,0)$, $(0,1,2,0)$ and $(0,0,3,4)$.

3. What does it mean to say that the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ are linearly independent?

   Let $\mathbf{A}$ be a linear operator on an n-dimensional vector space, having n distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ and corresponding eigenvectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$. Consider the action of the operator $\mathbf{A} - \lambda_i \mathbf{1}$ (where $\mathbf{1}$ is the identity operator) on the vector $\boldsymbol{e}_j$ in the cases $i = j$ and $i \neq j$. Hence, or otherwise, show that the vectors $\boldsymbol{e}_1, \boldsymbol{e}_2, \ldots, \boldsymbol{e}_n$ are linearly independent.

4. An $n \times n$ complex matrix A is such that each row and each column has exactly one non-zero element. The Hermitian conjugate of A is $\mathsf{A}^\dagger = (\mathsf{A}^{\mathrm{T}})^*$ (where A^{T} is the transpose of A, and A^* is its complex conjugate). Show that $\mathsf{A}^\dagger \mathsf{A}$ is a real diagonal matrix.

5. An Hermitian matrix A is one for which $\mathsf{A}^\dagger = \mathsf{A}$. Suppose that A and B are both Hermitian matrices. Show that $\mathsf{AB} + \mathsf{BA}$ is Hermitian. Also show that AB is Hermitian if and only if A and B commute.

6. Find the eigenvalues and eigenvectors of the matrix
$$\mathsf{A} = \begin{bmatrix} 1 & \alpha & 0 \\ \beta & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} ,$$
where neither of the complex constants α and β vanishes. Find the conditions for which (a) the eigenvalues are real, and (b) the eigenvectors are orthogonal. Hence show that both conditions are jointly satisfied if and only if A is Hermitian.

7. For an Hermitian matrix H, explain how to construct a unitary matrix U such that $\mathsf{U}^\dagger\mathsf{H}\mathsf{U} = \mathsf{D}$, where D is a real diagonal matrix. Illustrate the procedure with the matrix

$$\mathsf{H} = \begin{bmatrix} 4 & 3\mathrm{i} \\ -3\mathrm{i} & -4 \end{bmatrix} .$$

8. An anti-Hermitian matrix A is one for which $\mathsf{A}^\dagger = -\mathsf{A}$. What can be said about the eigenvalues of A?

   If S is real symmetric and T is real antisymmetric, show that $\mathsf{T} \pm \mathrm{i}\mathsf{S}$ are anti-Hermitian. Deduce that

$$\det(\mathsf{T} + \mathrm{i}\mathsf{S} - 1) \neq 0 .$$

   Show that the matrix

$$\mathsf{U} = (1 + \mathsf{T} + \mathrm{i}\mathsf{S})(1 - \mathsf{T} - \mathrm{i}\mathsf{S})^{-1}$$

   is unitary.

   For

$$\mathsf{S} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \mathsf{T} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

   show that the eigenvalues of U are $\pm(1 - \mathrm{i})/\sqrt{2}$.

9. Show that the eigenvalues of a real orthogonal matrix have unit modulus and that if λ is an eigenvalue then so is λ^*. Hence argue that the eigenvalues of a 3×3 real orthogonal matrix R must be a selection from

$$+1 , \quad -1 \quad \text{and} \quad \mathrm{e}^{\pm\mathrm{i}\alpha} .$$

   Verify that $\det \mathsf{R} = \pm 1$. What is the effect of R on vectors orthogonal to an eigenvector with eigenvalue ± 1?

10. Let H be an $n \times n$ Hermitian matrix with n distinct eigenvalues $\{\lambda_i\}$ and orthonormal eigenvectors $\{\mathsf{e}_i\}$. Now consider the slightly perturbed matrix $\mathsf{H} + \delta\mathsf{H}$, where $\delta\mathsf{H}$ is small and Hermitian. Let the eigenvalues and orthonormal eigenvectors of $\mathsf{H} + \delta\mathsf{H}$ be $\{\lambda_i + \delta\lambda_i\}$ and $\{\mathsf{e}_i + \delta\mathsf{e}_i\}$, respectively. By working to first order in the small quantities, show that

$$\delta\lambda_i = \mathsf{e}_i^\dagger(\delta\mathsf{H})\mathsf{e}_i ,$$

$$\delta\mathsf{e}_i = \sum_{j\neq i} \frac{\mathsf{e}_j^\dagger(\delta\mathsf{H})\mathsf{e}_i}{\lambda_i - \lambda_j}\, \mathsf{e}_j .$$

   Why is it permissible to omit any contribution to $\delta\mathsf{e}_i$ parallel to e_i? [Write out the eigenvector equation and the orthonormality condition for the eigenvectors of the perturbed matrix. Expand, neglect products of small quantities, and simplify. Use the fact that $\{\mathsf{e}_i\}$ is a basis.]

11. Find the eigenvalues and normalized eigenvectors of the symmetric matrices

$$\mathsf{A} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}, \qquad \mathsf{B} = \begin{bmatrix} 5 & 0 & \sqrt{3} \\ 0 & 3 & 0 \\ \sqrt{3} & 0 & 3 \end{bmatrix}, \qquad \mathsf{C} = \begin{bmatrix} 1 & 1 & 3 \\ 1 & 1 & -3 \\ 3 & -3 & -3 \end{bmatrix}.$$

Describe the related quadratic surfaces.

This example sheet is available at `http://www.damtp.cam.ac.uk/user/examples/`
Please send any comments and corrections to `M.Wingate@damtp.cam.ac.uk`