Example Sheet 1: Sturm-Liouville Theory and Variational Methods

1 Show that the equation

$$y'' + 2\gamma y' + (\gamma^2 + n^2)y = 0 \ ,$$

for constant γ, has non-zero solutions y_n that vanish at $x = 0$ and at $x = \pi$ when n is a positive integer. Put the equation into Sturm-Liouville form and hence find the function $w(x)$ such that

$$\int_0^\pi y_m(x) y_n(x) w(x) \ \mathrm{d}x = 0 \qquad \text{for } m \neq n \ .$$

2 Consider the fourth order differential operator

$$L = p(x)\frac{d^4}{dx^4} + q(x)\frac{d^3}{dx^3} + r(x)\frac{d^2}{dx^2} + s(x)\frac{d}{dx} + t(x) \ .$$

Find necessary conditions on p, q, r, s and t so that, for an inner product with a unit weight function

(i) L is self-adjoint;

(ii) $\mathcal{L} = w(x)L$ is self-adjoint, where $w(x)$ is a given function.

Can $w(x)$ always be chosen so that $\mathcal{L}$ is self-adjoint?

3 Find the eigenfunctions and eigenvalues of the differential operator

$$L = -\frac{d^2}{dx^2} + 1$$

acting on functions $y(x)$ subject to the boundary conditions $y(0) = y'(\pi) = 0$. Obtain the orthogonality relation for these eigenfunctions and write down Green's function for L. Expand $f(x) = x(x - 2\pi)$ in the eigenfunctions and hence obtain a solution of

$$y'' - y = x(x - 2\pi)$$

subject to the above boundary conditions.

4 Show that the operator

$$L = -(1-x^2)\frac{d^2}{dx^2} + 2x\frac{d}{dx}$$

is self-adjoint if both $y(1)$ and $y(-1)$ are required to be finite. Let $y_n = P_n(x)$ be the eigensolution to $Ly_n = \lambda_n y_n$ for $\lambda_n = n(n+1)$. Show that for $m \neq n$

$$\int_{-1}^{1} P_m(x)P_n(x)\mathrm{d}x = 0 \ , \qquad \text{and} \qquad \int_{-1}^{1}(1-x^2)\frac{dP_m}{dx}\frac{dP_n}{dx}\mathrm{d}x = 0 \ .$$

Given that $P_1 = x$ and $P_3 = \frac{1}{2}(5x^3 - 3x)$, find in terms of P_1 and P_3 the solution of the equation

$$Ly = x^3$$

with the above boundary conditions. What happens if x^3 is replaced by $x^2 - k$ where $k = \frac{1}{3}$?

Optional: discuss the case when $k \neq \frac{1}{3}$.

5 A coal box, in the shape of a cuboid, is to be placed flush against a wall, so that only its top, front and two sides are visible. The owner wishes the box to contain at least a certain volume V of coal, but also wishes to minimise the visible surface area. What lengths should be chosen for the sides?

6 The temperature within *and* on the surface of a sphere of unit radius is given by $T(x, y, z) = x(y + z)$. Find the minimum and maximum temperature.

7 Find the geodesics on a cylinder of radius a.

8 State Fermat's principle governing the paths traced by light rays and explain the conditions under which it applies. Given that in a horizontally stratified medium the refractive index is given by $\mu(z) = \sqrt{a - bz}$, where z is the height and a and b are positive constants, prove that light rays travelling in a vertical plane follow inverted parabolas. Show further that *all* such parabolas have their directrix in the plane $z = a/b$. [*The directrix of a parabola in standard form,* $y^2 = 4ax$, *is the line* $x = -a$.]

9 A particle of unit mass moves in a plane with polar coordinates (r, θ), under the influence of a central force derived from a potential $V(r)$. Write down the action functional for this problem and use Hamilton's principle to find differential equations for $r(t)$ and $\theta(t)$. Give a physical interpretation of these equations. Given that the particle's trajectory is $r = a\sin\theta$ for some constant a, deduce that (up to an arbitrary additive constant) $V \propto r^{-4}$.

10 If $\mu(\mathbf{r}) = |\nabla f(\mathbf{r})|$ for some function f, show that $\int_A^B \mu \, \mathrm{d}l$ between two points A and B is at least $f(B) - f(A)$, with equality if and only if the path of integration lies orthogonal to the family of surfaces $f =$ constant. Deduce that such orthogonal trajectories satisfy Fermat's principle.

11 A soap film is bounded by two circular wires at $r = a$, $z = \pm b$ in cylindrical polar coordinates (r, θ, z). Assuming that the soap surface is cylindrically symmetric, show that the equation of the surface of minimal area is

$$r = c \cosh(z/c)$$

where c satisfies the condition $a/c = \cosh(b/c)$. Show graphically that this condition has no solution for c if b/a is larger than a certain critical ratio. What happens to the soap surface as b/a is increased from below this ratio to above it?

12 An area is enclosed by joining two fixed points a distance a apart on a straight wall with a given length l of flexible fencing ($a < l < \pi a$). How is the area maximised?

13 Show from first principles that the equivalent of Euler's equation for the function $x(t)$ which extremises the integral

$$\int_{t_1}^{t_2} f(t, x, \dot{x}, \ddot{x}) \, \mathrm{d}t$$

with fixed values of both $x(t)$ and $\dot{x}(t)$ at $t = t_1$ and t_2 is

$$\frac{\partial f}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial f}{\partial \dot{x}}\right) + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\left(\frac{\partial f}{\partial \ddot{x}}\right) = 0.$$

Hence find the function $x(t)$ with $x(1) = 1$, $\dot{x}(1) = -2$, $x(2) = \frac{1}{4}$ and $\dot{x}(2) = -\frac{1}{4}$ that minimises $\int_1^2 t^4 \{\ddot{x}(t)\}^2 \, \mathrm{d}t$.

14 Consider the Sturm–Liouville problem

$$-(1 + x^2)y'' - 2xy' = \lambda y$$

with $y(\pm 1) = 0$. Use the Rayleigh–Ritz method to obtain an upper bound on the lowest eigenvalue by using the trial function $y_1 = 1 - x^2$. Show that a better bound is obtained from the trial function $y_2 = \cos(\pi x/2)$ and explain how a further improvement could be achieved by considering y_1 and y_2 in combination. [$\int_{-1}^{1} x^2 \sin^2(\pi x/2) \, dx = \frac{1}{3} + \frac{2}{\pi^2}$.]

15 The differential equation governing small transverse displacements $y(x)$ of a string with fixed end-points at $x = 0$ and $x = \pi$ is

$$y'' + \omega^2 f(x) y = 0$$

where ω is the angular frequency of the vibration and f is a positive function. Show that the allowed values of ω^2 are given by the stationary values of

$$\frac{\int_0^\pi y'^2 \, \mathrm{d}x}{\int_0^\pi f(x) y^2 \, \mathrm{d}x}.$$

Use this fact to find an approximate value for the angular frequency of the fundamental mode when $f(x) = 1 + \sin x$.

16 Show that $\psi_0 = \exp(-\frac{1}{2}x^2)$ is an eigenfunction of the operator

$$\mathscr{L} = -\frac{d^2}{dx^2} + (x^2 - 1)$$

acting on functions $\psi(x)$ for which $\psi \to 0$ as $|x| \to \infty$, and find the corresponding eigenvalue λ_0. This is in fact the lowest eigenvalue of the problem.

Use the Rayleigh–Ritz method with trial function

$$\widetilde{\psi}_0 = \begin{cases} b(a^2 - x^2) & |x| < a \\ 0 & |x| \geqslant a \end{cases}$$

where a and b are adjustable constants, to obtain the approximation

$$\widetilde{\lambda}_0 = \sqrt{10/7} - 1$$

to λ_0. Comment on the sign of $\widetilde{\lambda}_0 - \lambda_0$.

Comments on or corrections to this problem sheet are very welcome and may be sent to me at J.B.Gutowski@damtp.cam.ac.uk