Dynamics of Astrophysical Discs — Mathematical Tripos, Part III
Professor Gordon Ogilvie — Lent Term 2020

Solutions to the 2019 Tripos Paper

1.

(a) Potential in the midplane:

$$\Phi = -\frac{GM}{(r - 2R_\mathrm{g})}.$$

Angular velocity (assuming $r > 2R_\mathrm{g}$):

$$r\Omega^2 = -\Phi_r = \frac{GM}{(r - 2R_\mathrm{g})^2}$$

$$\Omega = \frac{1}{(r - 2R_\mathrm{g})}\sqrt{\frac{GM}{r}}.$$

Orbital shear parameter:

$$q = -\frac{d\ln\Omega}{d\ln r} = \frac{r}{(r - 2R_\mathrm{g})} + \frac{1}{2} = \frac{(3r - 2R_\mathrm{g})}{2(r - 2R_\mathrm{g})}.$$

Specific angular momentum:

$$h = r^2\Omega = \frac{\sqrt{GMr^3}}{(r - 2R_\mathrm{g})}.$$

Specific energy:

$$\varepsilon = \frac{1}{2}r^2\Omega^2 + \Phi = \frac{GMr}{2(r - 2R_\mathrm{g})^2} - \frac{GM}{(r - 2R_\mathrm{g})} = -\frac{GM(r - 4R_\mathrm{g})}{2(r - 2R_\mathrm{g})^2}.$$

Squared epicyclic frequency:

$$\Omega_r^2 = 2(2 - q)\Omega^2 = \frac{(r - 6R_\mathrm{g})}{(r - 2R_\mathrm{g})}\Omega^2 = \frac{(r - 6R_\mathrm{g})}{(r - 2R_\mathrm{g})^3}\frac{GM}{r}.$$

Orbits are unstable to horizontal perturbations when $\Omega_r^2 < 0$, i.e. when $r < 6R_\mathrm{g} = r_\mathrm{in}$.

Specific energy at inner radius:

$$\varepsilon_\mathrm{in} = -\frac{GM(6R_\mathrm{g} - 4R_\mathrm{g})}{2(6R_\mathrm{g} - 2R_\mathrm{g})^2} = -\frac{GM}{16R_\mathrm{g}} = -\frac{c^2}{16}.$$

(b) In a steady state

$$\mathscr{F} = \text{constant} = -\dot{M},$$

$$\mathscr{F} h + \mathscr{G} = \text{constant} = \mathscr{F} h_{\text{in}},$$

given that $\mathscr{G}_{\text{in}} = 0$. So

$$\mathscr{G} = -2\pi\bar{\nu}\Sigma r^3 \frac{d\Omega}{dr} = \dot{M}(h - h_{\text{in}})$$

$$\bar{\nu}\Sigma = \frac{f\dot{M}}{3\pi},$$

where

$$\begin{aligned} f &= \frac{3}{2q}\left(1 - \frac{h_{\text{in}}}{h}\right) \\ &= \frac{3(x-2)}{(3x-2)}\left[1 - \left(\frac{6}{x}\right)^{3/2} \frac{(x-2)}{4}\right] \\ &= \frac{(x-2)}{(x-\frac{2}{3})}\left[1 - \frac{3\sqrt{3}}{\sqrt{2}} \frac{(x-2)}{x\sqrt{x}}\right]. \end{aligned}$$

Within r_{in}, gas spirals rapidly into the black hole because the circular orbits are unstable. The radial velocity $|\bar{u}_r|$ increases rapidly inwards and so, to conserve mass, the surface density Σ declines rapidly. It is reasonable to assume that the low-density material inside r_{in} exerts a negligible torque on the disc.

Material accreted from large radius to the inner radius r_{in} loses $|\varepsilon_{\text{in}}| = \eta c^2$ in energy per unit mass. This is converted into heat and then radiation from the disc in a steady state, so the total luminosity of the disc is $L_{\text{disc}} = \eta \dot{M} c^2$, if advection of heat into the black hole can be neglected.

(c) The radiation pressure is

$$p_{\text{r}} = \frac{4\sigma T^4}{3c} = \left(\frac{\beta}{1+\beta}\right) p,$$

so

$$F_z = -\frac{c}{\kappa\rho}\frac{dp_{\text{r}}}{dz} = \frac{c}{\kappa}\left(\frac{\beta}{1+\beta}\right)\Omega^2 z$$

and

$$\begin{aligned}\frac{dF_z}{dz} = \rho\nu q^2\Omega^2 &= \frac{c}{\kappa}\left(\frac{\beta}{1+\beta}\right)\Omega^2 \\ \rho\nu &= \left(\frac{\beta}{1+\beta}\right)\frac{1}{q^2}\frac{c}{\kappa}.\end{aligned}$$

The vertically integrated viscosity is

$$\bar{\nu}\Sigma = \int \rho\nu\, dz = L_z\left(\frac{\beta}{1+\beta}\right)\frac{1}{q^2}\frac{c}{\kappa},$$

where L_z is the full vertical thickness. Equating this to $f\dot{M}/3\pi$ with

$$\dot{M} = \frac{\dot{M}}{\dot{M}_{\mathrm{E}}}\frac{4\pi GMc}{\kappa}\frac{1}{\eta c^2}$$

and $\eta = 1/16$ gives

$$\begin{aligned}L_z\left(\frac{\beta}{1+\beta}\right)\frac{1}{q^2}\frac{c}{\kappa} &= \frac{f}{3\pi}\frac{\dot{M}}{\dot{M}_{\mathrm{E}}}\frac{4\pi GMc}{\kappa}\frac{16}{c^2} \\ L_z &= \frac{64}{3}\left(\frac{1+\beta}{\beta}\right)q^2 f\frac{\dot{M}}{\dot{M}_{\mathrm{E}}}\frac{GM}{c^2},\end{aligned}$$

as required.

2.

(a) The Lagrangian is

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\phi}^2 + \dot{z}^2\right) - \Phi(r, z).$$

Define local coordinates (x, y, z) in the neighbourhood of an orbiting reference point:

$$r = r_0 + x, \qquad \phi = \Omega_0 t + \frac{y}{r_0}, \qquad z = z,$$

so that

$$L = \frac{1}{2}\left[\dot{x}^2 + (r_0 + x)^2\left(\Omega_0 + \frac{\dot{y}}{r_0}\right)^2 + \dot{z}^2\right] - \Phi(r_0 + x, z).$$

Expand L up to second order in the local coordinates:

$$L = L_0 + L_1 + L_2 + \cdots,$$

with

$$\begin{aligned}
L_0 &= \frac{1}{2}r_0^2\Omega_0^2 - \Phi_0, \\
L_1 &= r_0\Omega_0^2 x + r_0^2\Omega_0\dot{y} - \Phi_{r0}x, \\
L_2 &= \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \frac{1}{2}\Omega_0^2 x^2 + 2\Omega_0 x\dot{y} - \frac{1}{2}\Phi_{rr0}x^2 - \frac{1}{2}\Phi_{zz0}z^2,
\end{aligned}$$

where Φ_0, Φ_{r0}, Φ_{rr0}, etc., are Φ and its partial derivatives evaluated on the reference orbit at (r_0, z), and we use the property that Φ is odd in z. Thus

$$L_2 = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + 2\Omega_0 x\dot{y} - \Phi_{\mathrm{t}},$$

with tidal potential

$$\Phi_{\mathrm{t}} = \frac{1}{2}(\Phi_{rr0} - \Omega_0^2)x^2 + \frac{1}{2}\Phi_{zz0}z^2.$$

(b) Lagrange's equations of motion are

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = \frac{\partial L}{\partial q}$$

for $q = \{x, y, z\}$. Thus

$$\begin{aligned}
\ddot{x} &= 2\Omega_0\dot{y} - (\Phi_{rr0} - \Omega_0^2)x, \\
\ddot{y} + 2\Omega_0\dot{x} &= 0, \\
\ddot{z} &= -\Phi_{zz0}z.
\end{aligned}$$

In the case of a point-mass potential, we have

$$\begin{aligned}
\Phi &= -GM(r^2 + z^2)^{-1/2}, \\
\Phi(r, 0) &= -GMr^{-1}, \\
\Phi_r(r, 0) &= GMr^{-2}, \\
\Phi_{rr}(r, 0) &= -2GMr^{-3}, \\
\Phi_{zz0} &= GMr^{-3},
\end{aligned}$$

and $\Omega^2 = GMr^{-3}$, so we obtain

$$\begin{aligned}
\ddot{x} &= 2\Omega_0\dot{y} + 3\Omega_0^2 x, \\
\ddot{y} + 2\Omega_0\dot{x} &= 0, \\
\ddot{z} &= -\Omega_0^2 z.
\end{aligned}$$

So

$$p_y = \dot{y} + 2\Omega_0 x = \text{constant},$$

$$\ddot{x} + \Omega_0^2 x = 2\Omega_0 p_y,$$

with general solution

$$x = x_0 + \operatorname{Re}\left(A\,e^{-i\Omega_0 t}\right), \qquad x_0 = \frac{2p_y}{\Omega_0}.$$

Then

$$\begin{aligned}
\dot{y} &= p_y - 2\Omega_0 x_0 - 2\Omega_0 \operatorname{Re}\left(A\,e^{-i\Omega_0 t}\right) \\
y &= y_0 - \frac{3}{2}\Omega_0 x_0 t + \operatorname{Re}\left(-2iA\,e^{-i\Omega_0 t}\right),
\end{aligned}$$

and finally

$$z = \mathrm{Re}\left(B\, e^{-i\Omega_0 t}\right),$$

where x_0 and y_0 are arbitrary real constants and A and B are arbitrary complex constants.

As already noted, the conserved canonical y-momentum is

$$p_y = \frac{1}{2}\Omega_0 x.$$

The conserved energy is

$$\begin{aligned}
\varepsilon &= \dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}} - L \\
&= \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) + \Phi_{\mathrm{t}}.
\end{aligned}$$

This separates into two independent conserved quantities for the horizontal and vertical motion:

$$\varepsilon_{\mathrm{h}} = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{3}{2}x^2, \qquad \varepsilon_{\mathrm{v}} = \frac{1}{2}\dot{z}^2 + \frac{1}{2}z^2.$$

For our general solution these evaluate to

$$\begin{aligned}
\varepsilon_{\mathrm{h}} &= \frac{1}{2}\Omega_0^2(-A_{\mathrm{r}}\sin\Omega_0 t + A_{\mathrm{i}}\cos\Omega_0 t)^2 \\
&\qquad + \frac{1}{2}\left(-\frac{3}{2}\Omega_0 x_0 - 2A_{\mathrm{r}}\cos\Omega_0 t - 2A_{\mathrm{i}}\sin\Omega_0 t\right)^2 \\
&\qquad - \frac{3}{2}\Omega_0^2(x_0 + A_{\mathrm{r}}\cos\Omega_0 t + A_{\mathrm{i}}\sin\Omega_0 t)^2 \\
&= \frac{1}{2}\Omega_0^2\left(|A|^2 - \frac{3}{4}x_0^2\right), \\
\varepsilon_{\mathrm{v}} &= \frac{1}{2}\Omega_0^2(-B_{\mathrm{r}}\sin\Omega_0 t + B_{\mathrm{i}}\cos\Omega_0 t)^2 \\
&\qquad + \frac{1}{2}\Omega_0^2(B_{\mathrm{r}}\cos\Omega_0 t + B_{\mathrm{i}}\sin\Omega_0 t)^2 \\
&= \frac{1}{2}\Omega_0^2|B|^2,
\end{aligned}$$

as required.

(c) Collisions allow p_y to be exchanged between particles, while conserving the total. So the mean value $\langle x_0 \rangle$ over all particles remains zero and the ring remains centred on $x = 0$. Inelastic collisions cause the sum of $\varepsilon = \varepsilon_{\mathrm{h}} + \varepsilon_{\mathrm{v}}$ over all particles to decay. Initially $A = B = 0$ for all particles and ε is negative. As it decreases further, $\langle x_0^2 \rangle$ must increase (more so, because A and B become non-zero as a result of collisions). So the ring spreads symmetrically in the $\pm x$ directions.

3.

(a) Start from the ideal MHD equations as given, but including the Coriolis and tidal forces appropriate to the local model. Write $\mathbf{u} = -q\Omega x\,\mathbf{e}_y + \mathbf{v}$, where $\mathbf{v}$ is the departure from the orbital motion. Given that $\mathbf{v}$ and $\mathbf{B}$ depend only on z and t and that $v_z = 0$, we obtain $B_z =$ constant and

$$\begin{aligned}
\frac{\partial v_x}{\partial t} - 2\Omega v_y &= \frac{B_z}{\mu_0\rho}\frac{\partial B_x}{\partial z},\\
\frac{\partial v_y}{\partial t} + (2-q)\Omega v_x &= \frac{B_z}{\mu_0\rho}\frac{\partial B_y}{\partial z},\\
\frac{\partial B_x}{\partial t} &= B_z\frac{\partial v_x}{\partial z},\\
\frac{\partial B_y}{\partial t} + q\Omega B_x &= B_z\frac{\partial v_y}{\partial z},
\end{aligned}$$

as required. The q terms come from $\mathbf{v}\cdot\boldsymbol{\nabla}\mathbf{u}_0$ and $\mathbf{B}\cdot\boldsymbol{\nabla}\mathbf{u}_0$, where $\mathbf{u}_0 = -q\Omega x\,\mathbf{e}_y$. The Coriolis term $2\Omega\,\mathbf{e}\times\mathbf{u}_0$ cancels with the horizontal tidal force so that $\mathbf{v} = \mathbf{0}$ is a valid solution in the absence of $\mathbf{B}$.

(b) In a steady state:

$$\begin{aligned}
-2\Omega v_y &= \frac{B_z}{\mu_0\rho}\frac{dB_x}{dz},\\
(2-q)\Omega v_x &= \frac{B_z}{\mu_0\rho}\frac{dB_y}{dz},\\
0 &= B_z\frac{dv_x}{dz},\\
q\Omega B_x &= B_z\frac{dv_y}{dz},
\end{aligned}$$

so

$$\frac{d^2B_x}{dz^2} + K^2 B_x = 0, \qquad \frac{d^2B_y}{dz^2} = 0,$$

with

$$K^2 = \frac{2q\Omega^2\mu_0\rho}{B_z^2}.$$

With the given boundary conditions, the solution is

$$B_x = B_x^+ \frac{\sin(Kz)}{\sin(Kz^+)}, \qquad B_y = 0,$$

$$v_x = 0, \qquad v_y = -\frac{B_x^+ B_z}{2\Omega\mu_0\rho} \frac{K\cos(Kz)}{\sin(Kz^+)}.$$

(c) Introduce perturbations of the form

$$\delta v_x = \mathrm{Re}\left(\tilde{v}_x\, e^{st+ikz}\right),$$

etc., where $s \in \mathbb{C}$ is the growth rate and $k \in \mathbb{R}$ is the vertical wavenumber. The linearized equations require

$$\begin{aligned} s\tilde{v}_x - 2\Omega\tilde{v}_y &= \frac{B_z}{\mu_0\rho} ik\tilde{B}_x, \\ s\tilde{v}_y + (2-q)\Omega\tilde{v}_x &= \frac{B_z}{\mu_0\rho} ikB_y, \\ s\tilde{B}_x &= B_z\, ik\tilde{v}_x, \\ s\tilde{B}_y + q\Omega\tilde{B}_x &= B_z\, ik\tilde{v}_y. \end{aligned}$$

Multiply first two equations by ikB_z and use last two to substitute for v_x and v_y:

$$\begin{aligned} s^2\tilde{B}_x - 2\Omega(s\tilde{B}_y + q\Omega\tilde{B}_x) &= -\frac{k^2B_z^2}{\mu_0\rho}\tilde{B}_x, \\ s(s\tilde{B}_y + q\Omega\tilde{B}_x) + (2-q)\Omega s\tilde{B}_x &= -\frac{k^2B_z^2}{\mu_0\rho}\tilde{B}_y. \end{aligned}$$

Introduce the Alfvén frequency

$$\omega_\mathrm{a} = \mathbf{k}\cdot\mathbf{v}_\mathrm{a} = \frac{kB_z}{\sqrt{\mu_0\rho}},$$

so that

$$\begin{aligned} \left(s^2 + \omega_\mathrm{a}^2 - 2q\Omega^2\right) B_x - 2\Omega s B_y &= 0, \\ 2\Omega s B_x + \left(s^2 + \omega_\mathrm{a}^2\right) B_y &= 0, \end{aligned}$$

leading to the dispersion relation

$$\left(s^2 + \omega_\mathrm{a}^2 - 2q\Omega^2\right)\left(s^2 + \omega_\mathrm{a}^2\right) + 4\Omega^2 s^2 = 0,$$

as required.

The relevant solutions satisfying the boundary conditions ($\delta B_x = \delta B_y = 0$ at $z = \pm z^+$) have $B_x \propto B_y \propto \sin(kz)$ or $\cos(kz)$ with either $\sin(kz^+) = 0$ or $\cos(kz^+) = 0$, i.e. $kz^+ = n\pi/2$, $n = 1, 2, 3, \dots$.

Solve the quadratic for s^2, noting that $2(2-q)\Omega^2 = \Omega_r^2$:

$$s^4 + (2\omega_\mathrm{a}^2 + \Omega_r^2)s^2 + \omega_\mathrm{a}^2(\omega_\mathrm{a}^2 - 2q\Omega^2) = 0$$

$$s^2 = -\omega_\mathrm{a}^2 - \frac{\Omega_r^2}{2} \pm \sqrt{\frac{\Omega_r^4}{4} + 4\Omega^2\omega_\mathrm{a}^2}.$$

Instability ($s^2 > 0$ for $+$ root) if

$$\frac{\Omega_r^4}{4} + 4\Omega^2\omega_\mathrm{a}^2 > \left(\omega_\mathrm{a}^2 + \frac{\Omega_r^2}{2}\right)^2,$$

i.e.

$$\omega_\mathrm{a}^2(2q\Omega^2 - \omega_\mathrm{a}^2) > 0$$

(related to constant term in quadratic), i.e.

$$0 < \omega_\mathrm{a}^2 < 2q\Omega^2.$$

Since $\omega_\mathrm{a}^2 \propto k^2 \propto n^2$, the $n = 1$ mode is the critical one for overall stability. Equilibrium is unstable if

$$0 < \frac{k^2 B_z^2}{\mu_0\rho} < 2q\Omega^2$$

for $k = \pi/2z^+$, i.e. if

$$0 < \frac{\pi^2 B_z^2}{8q\mu_0\rho z^{+2}\Omega^2} < 1,$$

as required.

(d) The instability criterion is equivalent to

$$k^2 > \left(\frac{\pi}{2z^+}\right)^2,$$

where k is the wavenumber of the equilibrium solution. The first maximum of $B_x \propto \sin(kz)$ in $z > 0$ occurs at $z = \pi/2k$, which is less than z^+ if the disc is unstable.